Online Appendix

A Mechanism Design Approach to the Optimal Disclosure of Private Client Data

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This online appendix considers four distinct extensions of the original model. Each is covered in a different section. To facilitate the reading we separate the analysis from the proofs. In the first section we allow the buyer's interaction with S_1 to span over multiple periods. At the termination of this contract S_2 makes her take-it-or-leave it offer to the buyer, which we continue to assume that it is for a single trade in one period. We show that the latter's preferred signal is informative for almost any choice of correlations across periods and sellers, and we characterise this signal. A noteworthy exemption is when the buyer's type is static, in which case no disclosure is optimal.

The second section considers an alternative extension, which allows for the buyer's type to be continuous. We will not be able to derive the optimal signal for a generic distribution on the buyer's valuation and its evolution, however we will extend Calzolari and Pavan (2006) by providing some sufficient conditions for no disclosure to be optimal in our setup.

The third section shows that our analysis is still valid when moral hazard is introduced. Since this setting is mostly related to the labour market we will interpret the agent as an employee and the two principals as two employers. Hence the communication between the two employers has the natural interpretation of a reference letter.

The last section allows the termination time of the first contract to be specified by it, and provides some preliminary results on the optimal termination time.

1 Multi-period contracts

In this section we allow for S_1's interaction with the buyer to span over multiply periods, and explore the implications of this extension on her information provision problem. Assume that $t \in \{0, .., \infty\}$ and S_1's contract with the buyer is exogenously[1] terminated at the end of each period with probability $1-\gamma \in (0,1)$. Let $\tau \in \{0, .., \infty\}$ denote this exogenous termination time. Then the buyer's type θ_t evolves under S_1 according to

$$\begin{aligned} \varphi_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau > t) \\ \varphi_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau > t) \end{aligned}$$

whereas the transitioning probabilities across sellers are as in the baseline setup

$$\begin{aligned} \rho_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau = t) \\ \rho_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau = t) \end{aligned}$$

[1] This assumption is imposed mainly in order to simplify the notation. Even if termination was allowed to be endogenous, for any given choice of termination time S_1's information provision problem can be solved using the approach introduces in this section.

and we continue to denote by $\mu_0 = \Pr(\theta_0 = \theta_H)$ the initial public prior.

For the sake of simplicity it will be assumed that S_2 trades with the buyer only once in period $\tau + 1$, and then the game ends[2]. We will also assume that S_1 controls access to S_2, in the sense that the latter will make an offer only if the former's parternship with the buyer was exogenously terminated. However, we have argued in Appendix C that our results go through even without this assumption. We maintain the restriction that $\rho_H \geq \max\{\rho_L, \theta_L/\theta_H\}$, and add that $\varphi_H \geq \varphi_L$. Both principals can fully commit to their contracts. Conversely, the buyer will not be able to do so for any period in advance. As a result, the contract proposed by S_1 can essentially be interpreted as a series of history dependent single-period offers.

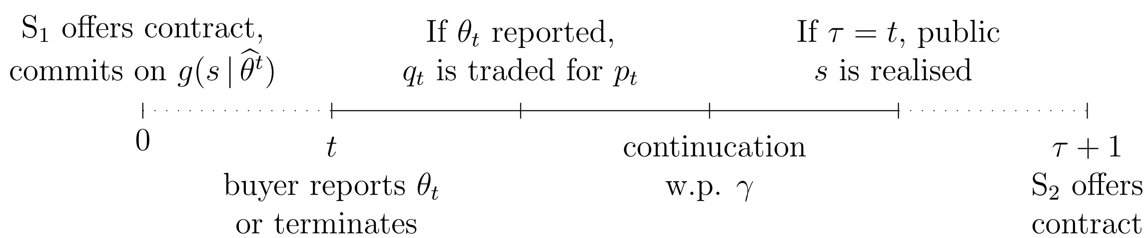
Hence, the timing on the multi-period model is as follows. At the beginning of period 1, S_1 publicly commits on a distribution $g_t(s \,|\, \widehat{\theta}^t)$, where $\widehat{\theta}^t$ denotes the history of reported types up to and including $\widehat{\theta}_t$, and offers to the buyer contract

$$\Big\{ \, p_t(\widehat{\theta}^t), \; q_t(\widehat{\theta}^t), \; g_t(s \,|\, \widehat{\theta}^t) \, \Big\}_{t=0}^{\infty}$$

which also includes a series of one period history depended prices and quantities. Subsequently, and at the beginning of each of the following periods, the buyer decides between either trading with S_1, which entails reporting his type, or terminating their partnership. Even if the buyer accept S_1's offer their contract may still end, after the quantity $q_t(\theta^t)$ has been traded, due to the exogenous termination probability $1 - \gamma$. If the contract is terminated at $\tau = t$ the s is realised and observed by S_2. Finally at $\tau + 1$, the period after its termination, S_2 offers to the buyer

$$\Big\{ \, p(\widehat{\theta}_{\tau+1}, s, \tau), \; q(\widehat{\theta}_{\tau+1}, s, \tau) \, \Big\}$$

which is a single period contract that utilises both the realised termination time τ and the signal s, and subsequently the game ends.



This section proceeds as follows. In subsection 1.1 we solve for S_2's optimal contract and characterise the buyer's post contractual payoff in this setting. In subsection 1.2 we derive a representation of S_1's payoff that does not depend on transfers and provide a generic sufficient condition for implementation. Next, in subsection 1.3 we show how the part of S_1's payoff that is related to the buyer's subsequent contract with S_2 can be reformulated so that her information design problem can be approached in a way similar to the literature of Bayesian Persuasion, and we characterise its solution.

[2] Equivalently, it could be assumed that the buyer's type is static under S_2. Dropping this restriction would not significantly alter the results of the main analysis, but it allows some useful closed-form expressions to be derived and it makes the connection between this and the two period model easier.

1.1 The buyer's post-contractual payoff

S_2's payoff maximisation problem is solved, and the buyer's expected payoff from trading with her is derived. For realised termination $\tau = t$, let $\mu_t^s = \Pr(\theta_t = \theta_H \,|\, s)$ denote S_2's posterior belief on the buyer's type in period t, and $\beta_t^s = \Pr(\theta_{t+1} = \theta_H \,|\, s)$ the corresponding posterior on θ_{t+1}. Those two are connected according to

$$\beta_t^s \;=\; \mu_t^s\, \rho_H \;+\; \left(1 - \mu_t^s\right) \rho_L$$

The problem itself is quite standard and its treatment can be found in the appendix. The following lemma characterises the buyer's payoff, which is the only result needed to proceed with S_1's payoff maximisation problem.

Lemma 1.1. *The payoff of a low buyer type under S_2 is equal to zero, while that of the high one equals*

$$B(\beta_t^s) \;=\; \begin{cases} b^{1+\epsilon} \cdot (\theta_H - \theta_L) \cdot \left(\frac{\theta_L - \beta_t^s \theta_H}{1-\beta_t^s}\right)^{\epsilon} & , \text{ if } \;\beta_t^s \le \theta_L/\theta_H \\ 0 & , \text{ if } \;\beta_t^s \ge \theta_L/\theta_H \end{cases} \tag{1.1}$$

Also, on the subset of posteriors $[0, \frac{\theta_L}{\theta_H})$ it is decreasing, and strictly concave (convex) for

$$\beta_t^s \;<(>)\; \frac{\theta_L}{\theta_H} + \frac{1-\epsilon}{2}\left(1 - \frac{\theta_L}{\theta_H}\right).$$

Proof. In Section 5. □

The payoff of a low buyer type in period 2 is always equal to zero, as it captures no rents. Conversely, the high type's payoff is positive, but only if the posterior β_t^s is low enough for S_2 to opt to serve both types. Indeed,we show in the proof that the rents he captures are proportional to the quantity bought by the low type. As β_t^s increases, a distortion on the low type's supplied quantity has smaller effect on S_2's expected payoff. Hence it becomes cheaper for her to increase this distortion in order to tighten the incentive compatibility constrain of the high type and decrease his rents. As a result, both the quantity supplied to the low type, and the high type's rents are decreasing in β_t^s.

1.2 Payoff equivalence

S_1's payoff maximisation problem is

$$\max_{p,q,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} \gamma^t \delta^t \cdot \Big(p_t(\theta^t) - c\big[q_t(\theta^t)\big] \Big) \right] \tag{$\mathcal{P}$}$$
$$\text{subject to IR}(\theta^t) \text{ and IC}(\theta^t)$$

where $\text{IR}(\theta^t)$ and $\text{IC}(\theta^t)$ refer to the individual rationality and incentive compatibility constrains of a θ^t buyer type. To make notation more compact three special cases of θ^t will be defined. First, let $\theta_L^t = \{\theta^{t-1}, \theta_L\}$ and $\theta_H^t = \{\theta^{t-1}, \theta_H\}$ denote a history such that the buyer's type in period t is low and high, respectively. In addition, for given generic θ^{t-1} and $t' \ge t$ let

$$L_t^{t'} \;=\; \{\theta^{t-1}, \theta_L, .., \theta_L\}\,, \tag{1.2}$$

denote a history such that the buyer's type has been low for all periods after, and including, period t. Also, whenever $t = 0$ simply write $L^{t'}$.

The proof of the subsequent proposition, which follows closely Battaglini (2005), demonstrates that the information rents captured by a period t high type θ_H^t are closely related to the histories $\{L_t^{t'}\}_{t'>t}$. In particular, when the implementation constrains, which will be provided shortly, do not bind the information rents captured by a period t high type are given by

$$U_t^H(\theta^{t-1}) \equiv \sum_{t'=t}^{\infty} \big[\gamma\delta(\rho_H - \rho_L)\big]^{t'-t} \cdot \Big\{(\theta_H - \theta_L) q_t(L_t^{t'}) \\ + \delta(1-\gamma)(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_{t'}^s) \,|\, L_t^{t'}\big]\Big)\Big\}$$

where $\beta_t^s = (\rho_H - \rho_L)\Pr(\theta_t = \theta_H \,|\, \tau = t, s) + \rho_L$. Then ($\mathcal{P}$) simplifies to the following problem, which only depends on policies (q, g) and not on transfers p.

Proposition 1.1. *Suppose that a solution of*

$$\max_{q,g} \; \Bigg\{ \mathbb{E}_\theta\Bigg[\sum_{t=0}^{\infty} \gamma^t\delta^t \cdot \Big(\theta_t\, q_t(\theta^t) - c\big[q_t(\theta^t)\big] \\ + \delta(1-\gamma)\Pr(\theta_{t+1} = \theta_H |\, \tau = t, \theta_t)\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta^t\big]\Big)\Bigg] \\ - \mu_0 \sum_{t=0}^{\infty} \gamma^t\delta^t(\varphi_H - \varphi_L)^t\Big[(\theta_H - \theta_L)q_t(L^t) + \delta(1-\gamma)(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_t^s) \,|\, L^t\big]\Big)\Big] \Bigg\} \quad (\mathcal{P}')$$

satisfies

$$(\theta_H - \theta_L)\Big[q_t(\theta_H^t) - q_t(\theta_L^t)\Big] + (\varphi_H - \varphi_L)\gamma\delta\Big[U_{t+1}^H(\theta_H^t) - U_{t+1}^H(\theta_L^t)\Big] \\ \geq \; (\rho_H - \rho_L)(1-\gamma)\delta\Big[\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_L^t\big] - \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big]\Big] \quad (\mathcal{P}_c)$$

Then those policies are also a solution to ($\mathcal{P}$) *and there exists a contract that implements them.*

Proof. In Section 5. □

The proof of the above proposition follows closely Battaglini (2005). ($\mathcal{P}_c$) is obtain by solving a relaxed version of ($\mathcal{P}$) where its downward slopping constrains IC(θ_L^t) are ignored, and showing that for this relaxed problem the upward slopping ones IC(θ_H^t) have to bind on its maximum. This allows the derivation of an expression for the period 0 high type's expected payments that only depends on the policies (q, g). The same can be done for the period 0 low type by using his individual rationality constrain. Substituting those expected payment in S_1's payoff gives ($\mathcal{P}'$). Henceforth, if the policies that solve the relaxed problem

($\mathcal{P}'$) satisfy the ignored downward slopping constrains IC(θ_L^t), then those also solve ($\mathcal{P}$). To check this, the derived expression for the expected payments is substituted in IC(θ_L^t), which gives ($\mathcal{P}_c$).

The expectation over the first summation in ($\mathcal{P}'$) denotes the ex-ante total surplus of S_1's partnership with the buyer, which includes the latter's post contractual payoff. Moreover, as shown in the proof of the proposition, the second summation denotes the rents captured by a period 1 high type. Note that even though a period t low type has some probability of becoming a high type, and as a result acquiring some rents in the future periods, S_1 can charge him in advance for those. Therefore, the buyer manages to capture positive rents only while he has never been a low type in the past. Interestingly, those rents are only related to the worst possible history $L^t = \{\theta_L, .., \theta_L\}$, which will create all the inefficiency of the multi-period contract.

For the model provided it can be shown that if the buyer's continuation value was not type depended, i.e. $\rho_H = \rho_L$, then the solution of ($\mathcal{P}'$) would always be implementable. However, this will not generically be true when $\rho_H > \rho_L$, since the high type has an additional incentive to misreport when the signal is informative. This is because the low type's signal will generically result in lower posteriors, which is beneficial for the buyer.

Corollary 1.1 (Implementation)**.** *The point-wise optimal level of production is*

$$q_t^*(\theta^t) = \begin{cases} (\theta_t)^\epsilon & , \ \theta^t \neq L^t \\ (\xi_t)^\epsilon & , \ \theta^t = L^t \end{cases} , \quad \textit{where} \ \ \xi_t = \max\left\{0\,,\, \theta_L - \frac{\mu_0(\theta_H - \theta_L)}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t\right\} \tag{1.3}$$

and this is implementable for any choice of signal distribution g if

$$\left(\frac{\theta_H}{\theta_L}\right)^\epsilon \ \geq \ 1 + \delta\left(\rho_H - \rho_L\right) b^{1+\epsilon} \tag{1.4}$$

Proof. In Section 5. □

The relaxed implementation condition (1.4) is identical to that of the two period model, and will hereafter be assumed to hold.

1.3 Information provision

This section solves S_1's information provision problem. Her optimal signal distribution g_t will be derived for any realisation of $\tau \in \{0, .., \infty\}$. Hence considering only the part of ($\mathcal{P}'$) that is affected by the signal s on realised $\tau = t$, and ignoring the discount factor δ^{t+1} that multiplies it, gives

$$\max_g \left\{ \sum_{\theta^t} \left[\Pr(\theta^t, \tau = t) \sum_s \Pr(\theta_{t+1} = \theta_H | \, \tau = t, \theta_t) B(\beta_t^s) g_t(s \,|\, \theta^t) \right] \right.$$
$$\left. - \, \mu_0 \Pr(\tau = t)(\varphi_H - \varphi_L)^t(\rho_H - \rho_L) \sum_s B(\beta_t^s) g_t(s \,|\, L^t) \right\} \tag{$\mathcal{G}_t$}$$

The first line of ($\mathcal{G}_t$) represents the expected rents from the buyer's contract with S_2, which are captured by S_1 through the individual rationality and incentive compatibility constrains of her own contract. On the other hand, the second line corresponds to the rents captured by the buyer in his first contract. Those are generated from the fact that whenever the signal s is informative the high buyer types θ_H^t have a stronger incentive to misreport, because their continuation payoff $B(\beta)$ is multiplied by ρ_H instead of ρ_L.

Next we transform ($\mathcal{G}_t$) into an equivalent problem that will only depend on the posteriors $\Pr(\theta_t = \theta_H \,|\, \tau = t)$ and $\Pr(\theta^t = L^t \,|\, \tau = t)$. Introduce the following notation

$$\begin{aligned} \mu_t &= \Pr(\theta_t = \theta_H \,|\, \tau = t), & \mu_t^s &= \Pr(\theta_t = \theta_H \,|\, s, \tau = t) \\ \lambda_t &= \Pr(\theta^t = L^t \,|\, \tau = t), & \lambda_t^s &= \Pr(\theta^t = L^t \,|\, s, \tau = t) \end{aligned}$$

The interim posterior beliefs on the first column only use the information provided by the termination time τ, whereas the posteriors on the second column also depend on the signal s; μ_t^s is the posterior on $\theta_t = \theta_H$, while λ_t^s on $\theta^t = L^t$. Note that the first event only depends on the contemporaneous θ_t, while the second on the whole history θ^t. Finally, abusing notation let $g_t(s)$ denote the probability of sending signal s after a contract being terminated at time $\tau = t$, that is

$$g_t(s) = \sum_{\theta_t} \Pr(\theta^t \,|\, \tau = t)\, g_t(s \,|\, \theta^t)$$

To proceed we provide the following Lemma, which generalises Lemma C.6 of Section C.2.

Lemma 1.2. *S_1's information provision problem in period t equivalently becomes*

$$\max_g \; \mathbb{E}_g[J_t(\mu_t^s, \lambda_t^s)] \tag{$\mathcal{G}_t$}$$

where its point-wise value $J_t(\mu_t^s, \lambda_t^s)$ is

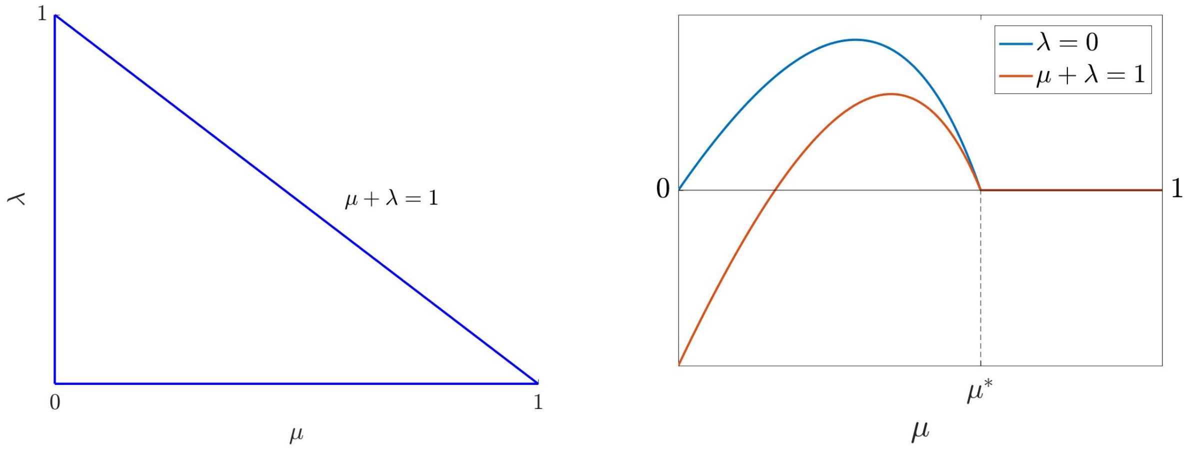
$$J_t(\mu_t^s, \lambda_t^s) = B(\beta_t^s)\big(\beta_t^s - \lambda_t^s \psi_t\big), \quad \text{and} \quad \psi_t \equiv \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t (\rho_H - \rho_L) \tag{1.5}$$

Proof. In Section 5. □

Similarly to the previous section, the information provision problem of S_1 is reformulated by considering (μ_t^s, λ_t^s) as the underline random variables, instead of the signal s, with joint distribution $\tilde{g}_t$. Hence, S_1 equivalently solves

$$\begin{aligned} \max_{\tilde{g}_t} \; \mathbb{E}_{\tilde{g}_t}\big[J_t(\mu, \lambda)\big] \quad \text{s.t. } & \mathbb{E}_{\tilde{g}_t}[\mu] = \mu_t, \quad \mathbb{E}_{\tilde{g}_t}[\lambda] = \lambda_t, \\ & \mu, \lambda \in [0,1], \text{ and } \mu + \lambda \le 1. \end{aligned} \tag{$\mathcal{G}'_t$}$$

The constrains ensure that the joint distribution $\tilde{g}_t(\mu, \lambda)$ is Bayes plausible. Note that it is not enough for μ and λ to be probabilities, that is to be in $[0,1]$, as they represent mutually exclusive events. Hence, their sum has to be less than one. The inequality can be strict as the history L^t does not necessarily cover all θ^t such that $\theta_t = \theta_L$. Therefore $\Pr(\theta_t \neq \theta_H \cap \theta^t \neq L^t)$, which is the complement of $\Pr(\theta_t = \theta_H \cup \theta^t = L^t) = \mu_t^s + \lambda_t^s$, can be strictly positive. Equivalently, it is possible that $\mu_t^s + \lambda_t^s < 1$.



(a) Domain (b) Boundaries

Figure 1: The domain of $J_t(\mu, \lambda)$, for $t > 1$, and its value on the two boundaries.

The domain of $J(\mu, \lambda)$ is a right-angled triangle, of which each of its legs has length one. A representative graph of it on the sides where $(\mu + \lambda = 1)$ and $(\lambda = 0)$ is given in Figure 1. Those two sides are connected by straight lines, as $J(\mu, \lambda)$ is linear in λ. It will be convenient to define $J_t(\mu, \lambda)$ as a function of of μ only on the two aforementioned boundaries. Hence, for $i \in \{f, t\}$ let

$$\bar{J}_i(\mu) \;=\; \zeta_i \, B(\beta)(\beta - \Psi_i)\,, \quad \text{where} \quad \begin{cases} \zeta_f = 1 & \Psi_f \;=\; 0 \\ \zeta_t \;=\; 1 + \frac{\psi_t}{\rho_H - \rho_L} & \Psi_t \;=\; \frac{\psi_t \rho_H}{\rho_H - \rho_L + \psi_t} \end{cases}$$

and note that $\beta = \mu\rho_H + (1 - \mu)\rho_L$ as those two posteriors are always connecting through this linear relationship. Lemma 5.2, which due to its size has being moved, together with the rest of the proofs, to Section 5 shows that $\bar{J}_f(\mu)$ represents the $(\lambda = 0)$ boundary, whereas $\bar{J}_t(\mu)$ the $(\mu + \lambda = 1)$ one. Interestingly, the latter's functional form is very similar to that of $J(\mu)$, derived in Section C.3, and that of the former is identical to $J_f(\mu)$, which was derived in Section C.2. This is because in the multi-period setup $J_f(\mu)$ corresponds to the representation of S_1's first best post contractual payoff.

To clarify the last claim suppose that S_1 could commit on (p, q, g) before learning the buyer's type, as in $(\mathcal{P})$, but his type was directly observed by her. Then instead of $J_t(\mu, \lambda)$, S_1 information provision problem would have $\bar{J}_f(\mu)$ as an objective function. This is because the rents paid by S_1 are multiplied by the probability of L^t to occur. Therefore $\lambda_t = 0$ corresponds to the case where no rents are paid, which is the same as ignoring the incentive compatibility constrains in $(\mathcal{P})$. In addition, for $\varphi_H < 1$ it is ease to show that $\bar{J}_f(\mu)$ is the limit of $J_t(\mu, \lambda)$ as t goes to infinity, which shows that S_1's information provision problem converges to its first-best solution after long contracting periods.

To solve S_1's information provision problem, first the concave closure of $J_t(\mu, \lambda)$ needs to be characterised. This will be denoted by $\mathcal{J}_t(\mu, \lambda)$ and defined as

$$\mathcal{J}_t(\mu, \lambda) \;=\; \sup\{z \,|\, (\mu, \lambda, z) \in \mathrm{co}(J_t)\},$$

where $\mathrm{co}(J_t)$ denotes the convex hull of the graph of $J_t(\mu, \lambda)$ on D.

In addition, for $i \in \{f,t\}$ let $\bar{\mathcal{J}}_i(\mu)$ denote the value of $\mathcal{J}_t(\mu,\lambda)$ on the subsets $(\lambda = 0)$ and $(\mu+\lambda = 1)$, respectively, which is also the concave closure of the corresponding $\bar{J}_i(\mu)$. The following proposition characterises $\mathcal{J}_t(\mu,\lambda)$. To facilitate its exposition, a new point needs to be introduced. For $i \in \{f,t\}$ and $\Psi_i < \frac{\theta_L}{\theta_H} < \rho_H$, let $\hat{\mu}_i$ be the unique solution of

$$\bar{J}_i(\hat{\mu}_i) + \bar{J}_i'(\hat{\mu}_i)(1-\hat{\mu}_i) \;=\; 0\,, \tag{1.6}$$

if this exists, and zero otherwise. The functional form of $\hat{\mu}_i$ can be found in the proof of the following proposition in, but it is not copied here, due to its size.

Proposition 1.2. *For any interior point $\mathcal{J}_t(\mu,\lambda) > J_t(\mu,\lambda)$.*

- *On the boundary $(\mu = 0)$: $\mathcal{J}_t(0,\lambda) = \bar{J}_t(0,\lambda)$.*
- *On the boundary $(\lambda = 0)$, and when $\Psi_t < \frac{\theta_L}{\theta_H}$ also on $(\mu+\lambda = 1)$:*

$$\bar{\mathcal{J}}_i(\mu) \;=\; \begin{cases} \bar{J}_i(\mu) & , \textit{for} \;\; \mu \;\leq\; \hat{\mu}_i \\ \bar{J}_i(\hat{\mu}_i) + \bar{J}_i'(\hat{\mu}_i)(\mu-\hat{\mu}_i) & , \textit{for} \;\; \mu \;\geq\; \hat{\mu}_i \end{cases} \tag{1.7}$$

- *On the boundary $(\mu+\lambda = 1)$, if $\Psi_t \geq \frac{\theta_L}{\theta_H}$ and $\epsilon \geq 1$, then $\mathcal{J}_t(\mu) = \bar{J}_t(\mu)$.*

Finally, $\hat{\mu}_i$ is non-decreasing in Ψ_i, and strictly increasing when it is positive.

Proof. In Section 5. □

The case where $\Psi_t \geq \theta_L/\theta_H$ and $\epsilon < 1$ can also easily be described, however it is omitted to make the statement of the above proposition more compact. Whenever the interim posteriors (μ_t,λ_t) are on one of the two boundaries, the optimal signal follows trivially from the corresponding bullet point. In particular, for the boundaries $(\lambda = 0)$ and $(\mu+\lambda = 1)$ the analysis is almost identical to that of Sections C.2 and C.3, respectively, hence it is omitted. On $(\mu = 0)$ any signal is optimal, informative or not, because $\mathcal{J}_t(0,\lambda)$ is linear on λ.

In addition, the above characterisation can be used to derive the optimal signal under static types

$$\varphi_H = 1-\varphi_L = \rho_H = 1-\rho_L = 1\,,$$

since in this case it has to be that $\mu_t + \lambda_t = 1$ for all periods $t \geq 0$. Non surprisingly the corresponding result of Section C.3, on the optimality of no information provision under static types, can be extended to the general model.

Corollary 1.2 (Privacy under Static Types)**.** *Suppose that the buyer's type is static, then no information provision is optimal on all periods $t \geq 0$.*

Proof. In Section 5. □

Unfortunately, it is not as ease to describe the optimal signal when (μ_t,λ_t) is an interior point, as the functional form of $\mathcal{J}_t(\mu,\lambda)$ on those point is harder to obtain. However, it is possible to provide the following characterisation.

Proposition 1.3. *For any interior point* (μ, λ) *the value of* $\mathcal{J}_t(\mu, \lambda)$ *is a linear combination between* $\bar{\mathcal{J}}_f(\mu') = \mathcal{J}_t(\mu', 0)$ *and* $\bar{\mathcal{J}}_t(\mu'') = \mathcal{J}_t(\mu'', 1-\mu'')$*, where*

$$\mu' = \mu - \frac{\lambda}{x}, \quad \textit{and} \quad \mu'' = \frac{1-\lambda+x\mu}{1+x}. \tag{1.8}$$

The values of $\mathcal{J}_t(\mu, \lambda)$ *and* x *are given by the solution of*

$$\begin{aligned} \mathcal{J}_t(\mu, \lambda) &= \max_x \left\{ (1-\mu-\lambda)\frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + \lambda\frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''} \right\} \\ & \quad s.t \quad x \in \left(-\infty, -\frac{1-\lambda}{\mu}\right] \cup \left[\frac{\lambda}{\mu}, +\infty\right) \end{aligned} \tag{1.9}$$

Proof. In Section 5. □

The proof relies on the fact that for any interior point (μ, λ) there exist two corresponding boundary points $(\mu_1, 0)$ and $(\mu_2, 1-\mu_2)$, along with an appropriate weight ω, such that $J_t(\mu, \lambda)$ can be written as a linear combination of those two points, that is

$$J_t(\mu, \lambda) = \omega J_t(\mu_1, 0) + (1-\omega) J_t(\mu_2, 1-\mu_2).$$

This makes it possible to write the concave closure $\mathcal{J}_t(\mu, \lambda)$ as a linear combination of its value on points of the boundaries $(\mu = 0)$ and $(\mu + \lambda = 1)$ exclusively. Therefore, $\mathcal{J}_t(\mu, \lambda)$ can be expressed as a linear combination of $\mathcal{J}_t(\mu', 0)$ and $\mathcal{J}_t(\mu'', 1-\mu'')$, the inputs of which belong in one of those two subsets of D. Hence, to find $\mathcal{J}_t(\mu, \lambda)$ it suffices to pick those two points by maximising the value of their linear combination, or equivalently pick the slope x of the line that connects the interior point (μ, λ) with the two boundaries. After some algebra it can be shown that this problem is equivalent to (1.9).

An immediate implication of Proposition 1.3 is that the optimal signal may need to use more than two, but no more than four possible realisations $s \in \{s'_0, s'_1, s''_0, s''_1\}$. This is because each of the values $\bar{\mathcal{J}}_f(\mu')$ and $\bar{\mathcal{J}}_t(\mu'')$ may require an additional linear combination on the corresponding boundary to be reached, similarly to the simple model of Section C.

Intuitively, S_1 engages in two distinct randomisations. First, she randomises over the boundary on which the posteriors will be, where $\bar{\mathcal{J}}_f$ denotes the first best, and $\bar{\mathcal{J}}_t$ the worst possible second best. This only affects the posterior $\lambda_t^s = \Pr(\theta^t = L^t)$. Subsequently, she randomises over the posterior $\mu_t^s = \Pr(\theta_t = \theta_H)$. It is noteworthy that S_2 only cares about μ_t^s, therefore the first randomisation has no effect on the buyer's expected post contractual payoff. In particular, in the first best S_1 would only provide information on the buyer's last reported type θ_t, and not on the history L^t. Despite that, in the second best the first randomisation over posteriors λ_t^s is used as a way to reduce the expected rents of a period 1 high type. Because those rents are mostly generated from the first periods, as t increases $\mathcal{J}_t(\mu, \lambda)$ converges to $\bar{\mathcal{J}}_f(\mu)$ and the effect of λ_t^s on S_1's post contractual payoff becomes negligible. Hence, her information provision problem converges to the first best, covered in Section C.2.

A closed-form representation of $\mathcal{J}_t(\mu, \lambda)$ is hard to obtain, since (1.9) may not have a corresponding closed-form solution, however the subsequent corollary identifies a case where this is possible.

Corollary 1.3. *Suppose that $\hat{\mu}_t = 0$, then*

$$\mathcal{J}_t(\mu, \lambda) \;=\; (1-\mu)\bar{J}_f(0) - \lambda\, \psi_t\,, \quad \textit{for all} \quad (\mu, \lambda) \in D\,. \tag{1.10}$$

Proof. As $\Psi_t > 0 = \Psi_f$, it follows from Proposition 1.2 that $\hat{\mu}_t = 0$ implies $\hat{\mu}_f = 0$. Hence substituting in (1.7) gives that for all $\mu \in [0, 1]$

$$\bar{\mathcal{J}}_f(\mu) \;=\; \bar{J}_f(\mu)(1-\mu) \quad \text{and} \quad \bar{\mathcal{J}}_t(\mu) \;=\; \bar{J}_t(\mu)(1-\mu)\,.$$

Then substituting the above in the objective function of (1.9) gives

$$\mathcal{J}_t(\mu, \lambda) \;=\; (1-\mu-\lambda)\, \bar{J}_f(0) + \lambda\, \bar{J}_t(0)$$

Finally, to obtain (1.10) substitute that $\bar{J}_t(0) = \bar{J}_f(0) - \psi_t$. □

Therefore, if $\mathcal{J}_t(\mu, \lambda)$ is linear on its boundaries, the same property holds on its interior. Because of this any randomisation between the two boundaries will define a corresponding optimal signal.

1.4 Special cases

This section considers S_1's information in two special cases. The first assumes that $\varphi_H = 1$, which implies that once the buyer becomes a high type he remains one. This specification is of interest because it describes the dynamics of information provision for addictive products, those that once the buyer develops a taste for them he becomes a loyal costumer. The second considers the generic case $\varphi_H < 1$ for very large t, that is it looks at the optimal information provision that follows the termination of contracts with very long duration.

1.4.1 Addictive products

To solve S_1's information provision problem under the restriction that $\varphi_H = 1$ note that for an buyer that is terminated in period t there are only two possible events; either $\theta_t = \theta_H$, or $\theta^t = L^t$. As a result it has to be that $\mu + \lambda = 1$. Hence, $J_t(\mu, \lambda) = \bar{J}_t(\mu)$. In addition, simple algebra shows that $\varphi_H = 1$ implies $\bar{J}_t(\mu) = \bar{J}_0(\mu)$ for all $t \geq 0$. This is identical to the objective function of S_1's information provision problem under single period contracts (C.14). However, the Bayes plausibility restriction is different since the prior on the buyer's type is μ_t, instead of μ_0, where for generic φ_L the former is given by

$$\mu_t \;=\; \mu_{t-1}(\varphi_H - \varphi_L) + \varphi_L \quad \Rightarrow \quad \mu_t \;=\; \mu_0(\varphi_H - \varphi_L)^t + \varphi_L\, \frac{1-(\varphi_H - \varphi_L)^t}{1-(\varphi_H - \varphi_L)}$$

As a result, under the restriction that $\varphi_H = 1$ the above becomes

$$\mu_t \;=\; 1 - (1-\mu_0)(1-\varphi_L)^t\,,$$

and S_1 equivalently solves

$$\max_{\tilde{g}_t} \; \mathbb{E}\big[\bar{J}_0(\mu)\big] \quad \text{s.t.} \;\; \mathbb{E}[\mu] = \mu_t\,, \quad \mu \in [0, 1]. \tag{1.11}$$

As argued in the previous section the effect of μ_0 on S_1's information provision problem is not clear, because it affects both the buyer's expected post contractual payoff and the information rents that a high type captures. Nevertheless, an increase on the posterior μ_t unambiguously bends S_1's optimal signal towards informativeness.

Proposition 1.4. *An informative signal is strictly optimal if and only if*

$$\max\left\{\rho_L, \rho_H\mu_0\right\} \;<\; \frac{\theta_L}{\theta_H} \quad \text{and} \quad \mu_t \;>\; \hat{\mu}_0\,, \tag{1.12}$$

in which case S_1's optimal signal $s \in \{\underline{s}, \overline{s}\}$ has distribution

$$g_t^*\left(\underline{s}\,|\,\theta_L\right) \;=\; 1\,, \qquad \text{and} \qquad g_t^*\left(\underline{s}\,|\,\theta_H\right) \;=\; \frac{1-\mu_t}{\mu_t}\frac{\hat{\mu}_0}{1-\hat{\mu}_0}. \tag{1.13}$$

Also, if an informative signal is strictly optimal for some t, then it remains so for all $t' \geq t$.

Proof. For $\rho_H\mu_0 < \theta_L/\theta_H$ the proof is identical to that of Proposition C.2. If $\rho_H\mu_0 \geq \theta_L/\theta_H$, then as argued in Section C.3 $\bar{J}_0(\mu)$ is flat for $\mu \geq \mu^*$ and negative below it. But then $\varphi_H = 1$ implies $\mu_t \geq \mu_0 \geq \mu^*$, hence $\mathcal{J}_0(\mu_t) = \bar{J}_0(\mu_t)$ for all $t \geq 0$ and information provision is never strictly optimal. The last statement follows trivially from noting that $\mu_{t+1} \geq \mu_t$. □

Intuitively, an increase in μ_t only affects S_1's post contractual payoff as it appears in the first best. This is because the rents captured by the high type depend on his initial reputation μ_0, but not on μ_t. To understand this note that a period 1 high type reveals all his private information in period 1, hence he captures no more rents in the subsequent periods. In addition, those are all the rents that S_1 pays, since if a period t high type had been a low one before, then she would have charged him for his future expected rents at this point.

Therefore, an increase in μ_t moves the buyer's post contractual payoff towards its flat part, which has the same effect on S_1's post contractual payoff, net of the rents paid to the period 1 high type. Hence, when μ_t becomes big enough S_1 may choose to randomise between revealing the high type and not. Since $\varphi_H = 1$ implies $\mu_{t+1} \geq \mu_t$, the above can also be restated in terms of the duration of the contract. That is the longer this duration is, the more likely S_1 becomes to provide an informative signal to S_2.

1.4.2 Long contracts

Next the restriction $\varphi_H = 1$ is dropped, but $t \to \infty$ is imposed instead. In particular, it will be assumed that $0 < \varphi_L \leq \varphi_H < 1$. In this case S_2's posterior on an buyer whose contract has not been terminated in period t is

$$\lim_{t\to\infty} \mu_t \;=\; \mu_\infty \;=\; \frac{\varphi_L}{1-(\varphi_H-\varphi_L)}\,,$$

In addition, it is ease to show that

$$\lim_{t\to\infty} J_t(\mu,\lambda) \;=\; \bar{J}_f(\mu)\,, \quad \text{and} \quad \lim_{t\to\infty} \Pr(\theta^t = L^t) \;=\; 0,$$

Hence the objective function of S_1's information provision problem becomes identical to that of single period contracts under the first best. However, the buyer's interim reputation is not his initial one μ_0, but its limit μ_∞.

Proposition 1.5. *For contracts of long duration, $t \to \infty$, if $\mu_\infty \leq \hat{\mu}_f$, then no information provision is optimal. In the opposite case, S_1's optimal signal $s \in \{\underline{s}, \overline{s}\}$, is informative and has distribution*

$$g_f\left(\underline{s} \,|\, \theta_L\right) \;=\; 1\,, \qquad \textit{and} \qquad g_f\left(\underline{s} \,|\, \theta_H\right) \;=\; \frac{1-\mu_\infty}{\mu_\infty}\frac{\hat{\mu}_f}{1-\hat{\mu}_f}. \tag{1.14}$$

Proof. Identical to that of Proposition C.1. □

As a result, both S_1's information provision problem and its solution are identical with that of the first best of the period t probem. The reason for that is that the amount of rents that S_1 pays for periods long in the future tends to zero. This is because, as argued before, those rents are all captured by a perpetually high type. However, the probability of facing such a type tends to zero as t goes to infinity. As a result, the expected amount of rents paid in period t also tends to zero, and the corresponding information provision problem becomes identical to the first best.

2 Continuous types

In this section we expand the model studied in Section C to allow for continuous types. Hence we consider again a two period model $t \in \{1, 2\}$, where in period 1 the buyer is offered a contract by S_1 and in period 2 by S_2. Instead of the binary type space of the previous sections, assume that θ_1 is distributed according to the continuously differentiable cumulative distribution function (CDF) $F_1(\theta_1)$ supported on $[\underline{\theta}_1, \overline{\theta}_1]$. Let $f_1(\theta_1) > 0$ denote the corresponding density. The inverse hazard rate of $F_1(\theta_1)$ is denoted by

$$\mu_1(\theta_1) \;=\; \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\,,$$

and similarly to most of the literature it is assumed that this is non-decreasing in θ_1. For the period 2 buyer type θ_2 suppose that its continuously differentiable CDF is $F_2(\theta_2 \,|\, \theta_1)$, with corresponding density $f_2(\theta_2 \,|\, \theta_1) > 0$ and support $[\underline{\theta}_2, \overline{\theta}_2]$. In addition, to capture some notion of positive correlation across periods assume that if $\tilde{\theta}_1 > \theta_1$, then $F_2(\cdot \,|\, \tilde{\theta}_1)$ first order stochastically dominates (FOSD) $F_2(\cdot \,|\, \theta_1)$. Finally, to simplify the exposition the support S of the signal s is restricted to be finite[3].

Similarly to the binary type specification, S_2's payoff maximisation problem is solved for any posterior $F_2^s(\theta_2)$. Let the corresponding inverse hazard rate be denoted by

$$\mu_2^s(\theta_2) \;=\; \frac{1-F_2^s(\theta_2)}{f_2^s(\theta_2)}$$

Lemma 2.1. *The payoff of a θ_2 buyer type from his contract with S_2 is*

$$V_2^s(\theta_2) \;=\; b\int_{\underline{\theta}_2}^{\theta_2} q_2^s(x)\,\mathrm{d}x\,, \tag{2.1}$$

[3]This rules out perfect revelation of θ_1, however the statements that will be made on the optimality of no information provision would also hold if such a signal was allowed. More generically, the subsequent analysis holds for any CDF $G(s \,|\, \theta_1)$, with support $[\underline{s}, \overline{s}]$, that induces integrable posteriors.

where $q_2^s(\theta_1)$ is the solution of S_2's payoff maximisation problem

$$\max_{q_2} \int_{\underline{\theta}_2}^{\bar{\theta}_2} \Big\{ b\, q_2(\theta_2)\big[\theta_2 - \mu_2^s(\theta_2)\big] - c[q_2(\theta_2)] \Big\} \,\mathrm{d}F_2^s(\theta_2), \qquad (2.2)$$
$$\textit{subject to} \;\; q_2(\theta_2) \;\; \textit{being non-decreasing.}$$

Proof. In Section 6. □

Next, the focus is turned to S_1's payoff maximisation problem, part of which is the choice of $g(s \,|\, \theta_1)$. For a given signal realisation s, let the expected payoff a θ_1 buyer type from his contract under S_2 be denoted by

$$\bar{V}_2^s(\theta_1) \;=\; \mathbb{E}_{\theta_1}\big[V_2^s(\theta_2) \,|\, \theta_1, s\big]$$

and note that this is not a function of the reported $\hat{\theta}_1$. Nevertheless, this influences the buyer's post contractual payoff $\mathbb{E}_g\big[\bar{V}_2^s(\theta_1) \,|\, \hat{\theta}_1\big]$, since the distribution of s potentially depends on $\hat{\theta}_1$. To connect this with the analysis of the binary type space, I first look at S_1's information provision problem under the first best.

Proposition 2.1. *Suppose that the buyer's type is static and that there are some types that are not supplied by S_2 under no information provision, equivalently $\mu_1(\underline{\theta}_1) > \underline{\theta}_1$. Then S_1's first best contract entails some information provision.*

Proof. In Section 6. □

The proof follows from a simple observation, whenever some buyer types are not contracted by S_2 this is inefficient for S_1. This is because they do not capture any rents from S_2's contract. The reason why S_2 would choose to exclude some buyer types from her contract is because the asymmetry of information can potentially be significant enough so that supplying them would be non-profitable. But then S_1 can reduce this asymmetry by reporting whenever the buyer's type belongs in this set of excluded types. This does not affect the rents captured by the higher types, since the excluded types would not be part of their contract to begin with. However, some of the revealed low types will be supplied a positive quantity, since the asymmetry of information will be reduced.

Depending on the value of ϵ similar arguments can be used even if all buyer types were contracted by S_2, however in this case it would be the convexity of her supply schedule that would be make randomisation between revealing information and not profitable for S_1. Note that even though the above argument is given for static types, its underline intuition is still relevant under dynamic ones. However, in this case more structure needs to be imposed on $F_2(\theta_2 \,|\, \theta_1)$, as for example is done later in this section when the second best is discussed.

Now, the focus of the analysis is switched back to the second best under S_1. The payoff of a period 1 buyer of type θ_1 when reporting $\hat{\theta}_1$ is

$$\widehat{V}_1(\hat{\theta}_1, \theta_1) \;=\; \theta_1 q_1(\hat{\theta}_1) - p_1(\hat{\theta}_1) + \mathbb{E}_g\big[\bar{V}_2^s(\theta_1) \,|\, \hat{\theta}_1\big]\,.$$

Let the buyer's payoff under truthful reporting be $V_1(\theta_1) = \widehat{V}_1(\theta_1, \theta_1)$. Then S_1's revenue maximisation problem is

$$\max_{p_1, q_1, g} \int_{\underline{\theta}_1}^{\bar{\theta}_1} \Big\{ p_1(\theta_1) - c[q_1(\theta_1)] \Big\} \, \mathrm{d}F_1(\theta_1)\,, \\ \text{subject to } \; V_1(\theta_1) \;=\; \max_{\hat{\theta}_1} \widehat{V}_1(\hat{\theta}_1, \theta_1) \tag{2.3}$$

Let $F_1^s(x) \equiv \Pr(\theta_1 \leq x \,|\, s)$ denote the posterior CDF on θ_1 after a signal realisation $s \in S$, with corresponding inverse hazard rate

$$\mu_1^s(\theta_1) = \frac{1 - F_1^s(\theta_0)}{f_1^s(\theta_1)}$$

Lemma 2.2. *S_1's information provision problem is*

$$\max_{g} \; \mathbb{E}_g \left[\int_{\underline{\theta}_1}^{\bar{\theta}_1} \Big\{ \mu_1^s(\theta_1) - \mu_1(\theta_1) \Big\} \frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} \, \mathrm{d}F_1^s(\theta_1) \right] \tag{2.4}$$

A sufficient condition for $g(s \,|\, \theta_1)$ to be implementable along with the point-wise optimal choice of production $q_1^*(\theta_1) = \max\big\{0\,,\, \theta_1 - \mu_1(\theta_1)\big\}^{\epsilon}$ is that

$$q_1(\hat{\theta}_1) + \mathbb{E}_g \left[\frac{\mathrm{d}\bar{V}_2(\theta_1)}{\mathrm{d}\theta_1} \,\middle|\, \hat{\theta}_1 \right] \tag{2.5}$$

is non-decreasing in $\hat{\theta}_1$. This always holds under no information provision.

Proof. In Section 6. □

Next, the optimal deterministic signalling structure is derived. That is the optimal among the ones where S_1 does not randomise. On this subset of distributions $g(s \,|\, \theta_1)$, she is restricted to choosing a partition $\{\Theta_1^s\}_{s \in S}$ of $[\underline{\theta}_1, \bar{\theta}_1]$ and reporting to S_2 the set of this partition on which θ_1 belongs. Hence the realised signal s satisfies $\theta_1 \in \Theta_1^s$. Interestingly, it is relatively ease to show that under such a signalling structure $\mu_1^s(\theta_1) \leq \mu_1(\theta_1)$, which combined with (2.4) gives the following.

Proposition 2.2. *Suppose that S_1 is restricted to using deterministic signals, then no information provision is optimal.*

Proof. In Section 6. □

The value of this proposition is twofold. First, it is ease to imagine scenarios where S_1 will not be able to credible randomise, but she can still reveal some information on the buyer's types. Such cases can be modelled by restricting attention to deterministic signals. In addition, the above result demonstrates that if S_1 cannot credible randomise, then there are no benefits from establishing some other more restrictive device of communication. The second reason why the above result is important is related to the technical complexity of calculating the optimal signal without this restriction.

In the remaining of this section, a case is presented for which S_1's optimal signal can be derived. Hence drop the restriction of deterministic signals, and instead impose the following specification on $F_2(\theta_2 \,|\, \theta_1)$. For the type of period 2 assume that

$$\begin{cases} \theta_2 = \theta_1 & \text{, with probability} \quad \rho \\ \theta_2 \sim F_1(\cdot) & \text{, with probability} \quad 1-\rho \end{cases}$$

This structure allows for both perfect correlation, which is equivalent to static types, and full independence. Intuitively, somebody would expect that information provision is optimal for interior values of ρ, as in the binary type specification, however the opposite is true.

Proposition 2.3. *Suppose that the buyer's type is redrawn with probability ρ, then no information provision is optimal for all $\rho \in [0,1]$.*

Proof. In Section 6. □

The above proposition demonstrates that just imperfect correlation is not enough for information provision to be optimal. Its proof underlines a novel result, which builds on previous work from Calzolari and Pavan (2006). This is that if in the absence of information provision S_2 opts for the same quantities that S_1 would if she was integrated with her, then no information provision is optimal. In some sense, disclosure has value only if it moves the policy choices away from those that maximise the total surplus of both S_1 and S_2. As shown in the appendix this is not true under the above specification of $F_2(\theta_2 \,|\, \theta_1)$.

3 Moral Hazard, Employment Contracts, and References

In this section we consider an alternative version of our mutli-period model, in which we allow for both moral hazard and endogenous termination. Continue to assume that $t \in \{0, ..., \infty\}$. Also because this setting is more often associated with the labour market we will switch to having two principals P_a and P_b (she) interacting with a single agent (he). All three of them are risk neutral, and discount the future with $\delta \in (0,1)$. At time zero P_a proposes a contract to the agent. If accepted, it lasts up to a termination time τ. The next period, i.e. $\tau + 1$, P_a switches to her outside option and the agent receives a new offer from P_b. For simplicity it is assumed that P_b approaches the agent only if he first entered a contract with P_a. All outside options are normalised to zero. If an agent is in a contract with one of the principals in period t, then the value of his production is

$$y_t^a = \theta_t \, e_t \quad \text{and} \quad y_t^b = b \, \theta_t \, e_t$$

for P_a and P_b, respectively. On the other hand, the agent's period payoff is

$$w - c(e_t), \quad \text{where} \quad c(e_t) = \frac{(e_t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}$$

and $e_t \geq 0$ is the effort level, which the agent chooses privately. The agent's ability $\theta_t \in \{\theta_L, \theta_H\}$, $0 < \theta_L < \theta_H$ is also his private information. The public prior in period 0 is

$\mu_0 = \Pr(\theta_0 = \theta_H) \in (0,1)$. During the agent's employment under P_a his type θ_t evolves stochastically according to

$$\begin{aligned}\varphi_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau > t) \\ \varphi_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau > t)\end{aligned}$$

In contrast, θ_t will be assumed to be perfectly sticky[4], that is $\Pr(\theta_{t+1} = \theta_t \,|\, t > \tau) = 1$. However, the agent's type will be allowed to evolve between the two principals according to

$$\begin{aligned}\rho_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau = t) \\ \rho_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau = t)\end{aligned}$$

Similarly to before we assume that $\varphi_H \geq \varphi_L$ and $\rho_H \geq \rho_L$.

We allow for P_a's partnership with the agent to be terminated in two possible ways. The first possibility is that it is exogenously severed, which is assumed to occur with probability $1-\gamma$ at the end of each period. The second possibility is that it is terminated endogenously, that is the contract offered by P_a specifies its termination after a certain set of histories. Let τ_γ denote the exogenous termination time and τ_a the endogenous one. Then the realised termination time is

$$\tau = \min\{\tau_\gamma, \tau_a\}$$

Two distinct cases will be considered for τ_a. The first will restrict P_a to offering a contract that sets $\tau_a = \infty$. This type of contract can only be terminated from the exogenous τ_γ, and have two interesting subcases. For $\gamma = 0$ it becomes a fixed term contract, as it is always terminated at the end of period 0. On the other hand, for $\gamma > 0$ it resembles a tenure contract, because it is only halted due to exogenous circumstances. Finally, the second case that will be considered will allow P_a to commit ex-ante on a history depended τ_a, which will result on flexible contracts. Define the probability of endogenous continuation

$$f_t(I_t) \;=\; \Pr(\tau_a > t \,|\, \tau_a > t-1 \,,\, I_t),$$

where I_t denotes the information set available to P_a at the end of period t. That is $f_t(I_t)$ is the probability that the agent's contract with P_a will continue on $t+1$, endogenously at least, given that it was active on t. Note that if the agent is paired with P_a In period t, that implies that the contract was not terminated at the end of the previous period, which is why the probability is conditioned on $\tau_a > t-1$. As a result the probably of continuation is $\gamma f_t(I_t)$.

The interaction between P_a and the agent will be partly private. On the one hand, P_b will learn the realised termination time τ and the contract offered by P_b. On the other, she will observe neither the realised production $\{y_t^a\}_{t=0}^{\tau}$, nor the agent's reports to P_a's mechanism[5]. However, P_a will be assumed to be able to credible convey additional information to P_b by committing ex ante on a signal $s \in S$ with distribution $g_t(s \,|\, I_\tau)$.

[4]This restriction is imposed mainly to facilitate the exposition. The main results would not be affected if it was dropped.

[5]Even if P_a's mechanism was private, it would be without loss to focus on equilibria where she credible reveals it to P_b.

Both principals can fully commit on their contracts, and the agent is not subject to limited liability. To avoid unnecessary complexity it will be assumed that even though the agent cannot commit on not leaving P_a's contract on $t < \tau$, if he chooses to do so the latter can make sure that he will not receive an offer from P_b[6] . Let the history of realised types up to t be denoted by $\theta^t = \{\theta_0, .., \theta_t\}$. It is ease to show that the revelation principle applies in this setting, thus let $\hat{\theta}^t$ denote the history of the agent's reported types. At time zero P_a offers to the agent the following contract

$$\left\{\, w^a_t(\hat{\theta}^t, y_t),\; e^a_t(\hat{\theta}^t),\; f_t(\hat{\theta}^t),\; g_t(s \,|\, \hat{\theta}^t) \,\right\}_{t=0}^{\infty},$$

which specifies his compensation, the recommended effort level, the contract's termination time, and the signal's conditional distribution, respectively. After the agent's employment under P_a has being terminated, and only if he accepted her proposal, P_b offers contract

$$\{\, w^b(\hat{\theta}_{\tau+1}, \tau, s, y^b_t),\; e^b(\hat{\theta}_{\tau+1}, \tau, s) \,\}.$$

where $\hat{\theta}_{\tau+1}$ is the agent's report on his valuation of P_b's product. In this case, it is without loss to consider a static contract, because θ_t does not fluctuate for $t \geq \tau + 1$. Moreover, this depends on the termination time τ and the realised signal s because both affect P_b's posterior.

3.1 Payoff Equivalence and Implementation

In this subsection, first the agent's payoff under P_b is derived. Second, a representation of P_a's payoff is obtained that does not depend on wages. Third, a generic sufficient condition for implementation is provided. Forth, it is shown how the part of P_a's payoff that is related to the agent's subsequent contract with P_b can be reformulated so that the information design problem of P_a can be approach in a way similar to the literature on Bayesian Persuasion.

3.1.1 The agent's post-contractual payoff

P_b is facing a simple static mechanism design problem with binary types. For realised termination time t let $\beta^s_t = \Pr(\theta_{t+1} = \theta_H \,|\, \tau = t\,, s)$ denote her posterior belief on θ_{t+1}, which is connected to her posterior on θ_t according to

$$\beta^s_t \;=\; \rho_H \Pr(\theta_t = \theta_H \,|\, \tau = t\,, s) + \rho_L \Pr(\theta_t = \theta_L \,|\, \tau = t\,, s).$$

P_b payoff maximisation problem is quite standard and its treatment can be found in the appendix. The following lemma characterises the agent's continuation payoff, which is the only result relevant to P_b's problem. To simplify its statement define the following two parameters

$$\kappa = \left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} \quad \text{and} \quad K \;=\; \frac{1-\kappa}{1-\delta}\,\frac{(b\cdot\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}}.$$

[6] Alternatively, the contract would also have to specify the information provided on such an even, even though it would never happen on path. P_a would find it optimal to always allocate some non-zero probability of revealing a high type, which is the agent's least preferred signal, so that she can punish early departures. This would also increase the agent's outside option, which would decreases P_a payoff by a fixed constant. However, it would not alternate her optimal contract, as long as this was implementable.

Lemma 3.1. *The total discounted payoff of a low agent type under P_b is always equal to zero, while that of the high one is equal to*

$$B(\beta_t^s) \;=\; K \cdot \left(\frac{1-\beta_t^s}{1-\beta_t^s\,\kappa}\right)^{1+\epsilon} \tag{3.1}$$

which is a strictly decreasing function.

Proof. In Section 7. □

The continuation payoff of a low agent type is always equal to zero, as he captures no rents. In contrast, the high type's payoff is positive, but decreasing and for $\beta_t^s = 1$ it actually becomes zero. Intuitively, the more likely a high type becomes, the less a distortion on the low type's production affects P_b's expected payoff. Hence it becomes cheaper for P_b to use this distortion to incentivise the truthful reporting of θ_H.

3.1.2 Payoff Equivalence

The revelation principle applies for P_a's mechanism design problem. Moreover, using the reported type, she can construct a perfect estimate of the agent's choice of effort. Hence, any misalignment between this estimate and the recommended effort can be punished strongly enough for the agent to have to mask it. As a result, a history of reports $\hat{\theta}^t$ implies choice of effort

$$\hat{e}_t^a(\hat{\theta}^t, \theta_t) \;=\; e_t^a(\hat{\theta}^t) \cdot \frac{\hat{\theta}_t}{\theta_t}$$

Hereafter, y_t will be dropped from the on path wage $w(\hat{\theta}^t, y_t)$, because this will not depend on it. Let the probability that the contract will endogenously continue up to t', conditional on not have being endogenously terminated at $t-1$, and the history $\theta^{t'}$ be denoted by

$$f_t^{t'}(\theta^{t'}) \;=\; \begin{cases} 1 & \text{, for } \; t' < t \\ \Pr(\tau_a > t' \,|\, \tau_a > t-1\,,\, \theta^{t'}) & \text{, for } \; t' \geq t \end{cases}$$

A special case of this is $t' = t$, where it becomes the probability of endogenous continuation $f_t(\theta^t)$. Using the above notation P_a's payoff maximisation problem becomes

$$\max_{w,e,f,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t\Big(\theta_t\, e_t^a(\theta^t) - w_t^a(\theta^t)\Big)\right] \tag{$\mathcal{P}$}$$
$$\text{subject to IR}(\theta^t) \text{ and IC}(\theta^t),$$

where IR(θ^t) and IC(θ^t) refer to the individual rationality and incentive compatibility constrains, respectively, of a θ^t agent type. Note that continuing up to period t only depends on history θ^{t+1}, as the decision to not terminate the contract is taken at the end of each period, with the last relevant devision being in period t.

To make notation more compact three special cases of θ^t will be defined. First, let $\theta_L^t = \{\theta^{t-1}, \theta_L\}$ and $\theta_H^t = \{\theta^{t-1}, \theta_H\}$ denote a history such that the buyer's type in period t is low and high, respectively. In addition, for given generic θ^{t-1} and $t' \geq t$ let

$$L_t^{t'} \;=\; \{\theta^{t-1}, \theta_L, .., \theta_L\}\,, \tag{3.2}$$

denote a history such that the buyer's type has been low for all periods after, and including, period t. Also, whenever $t=0$ simply write $L^{t'}$.

The proof of the subsequent proposition follows closely Battaglini (2005) and it is almost identical to that of Proposition 1.1, demonstrates that the information rents captured by a period t high type θ_H^t are closely related to the histories $\{L_t^{t'}\}_{t'>t}$. In particular, when the implementation constrains, which will be provided shortly, do not bind the information rents captured by a period t high type are given by

$$U_t^H(\theta^{t-1}) \;=\; \sum_{t'=t}^{\infty} f_t^{t'-1}(L_t^{t'-1})[\gamma\delta(\varphi_H-\varphi_L)]^{t'-t}\Bigg\{\frac{e_t^a(L_t^{t'})^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa) \\ +[1-f_{t'}(L_t^{t'})\gamma](\rho_H-\rho_L)\delta\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,L_t^{t'}\big]\Big)\Bigg\}$$

Then ($\mathcal{P}$) simplifies to the following problem, which only depends on policies (e_t^a, f_t, g_t) and not on the price p_t^a, paid by the agent to P_a.

Proposition 3.1. *Suppose that a solution of*

$$\max_{e,f,g}\;\Bigg\{\mathbb{E}_\theta\Bigg[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t\Big(\theta_t e_t^a(\theta^t)-\frac{e_t^a(\theta^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \\ +[1-f_t(\theta^t)\gamma]\delta\Pr(\theta_{t+1}=\theta_H|\,\tau=t,\theta_t)\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta^t\big]\Big)\Bigg] \\ -\mu_0\sum_{t=0}^{\infty} f_0^{t-1}(L^{t-1})\gamma^t\delta^t(\varphi_H-\varphi_L)^t\left[\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa)+\delta[1-f_t(L^t)](\rho_H-\rho_L)\mathbb{E}_g\big[B(\beta_t^s)\,|\,L^t\big]\Big)\right]\Bigg\} \tag{$\mathcal{P}'$}$$

satisfies

$$\frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{1}{\kappa}-1\right)-\frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa)+(\varphi_H-\varphi_L)\gamma\delta\Big[f_t(\theta_H^t)U_{t+1}^H(\theta_H^t)-f_t(\theta_L^t)U_{t+1}^H(\theta_L^t)\Big] \\ \geq\;(\rho_H-\rho_L)\delta\Big[\big[1-f_t(\theta_L^t)\gamma\big]\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_L^t\big]-\big[1-f_t(\theta_H^t)\gamma\big]\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big]\Big] \tag{$\mathcal{P}_c$}$$

Then those policies are also a solution to ($\mathcal{P}$) and there exists a contract that implements them.

Proof. In Section 7. □

The above representation is obtain by ignoring the downward slopping IC constrains, i.e. $\mathrm{IC}(\theta^{t-1},\theta_L)$, of ($\mathcal{P}$) and showing that in this case the upward slopping ones, that is $\mathrm{IC}(\theta^{t-1},\theta_H)$, bind. This allow the derivation of an expression for the period 0 high type's wages that only depends on the policies (e,τ,g). The same can be done for the period 0 low type by using its individual rationality constrains. Substituting those in P_a's payoff gives ($\mathcal{P}'$). As a result, whenever its solution satisfies the ignored constrains this is also a solution to ($\mathcal{P}$). In order to check that the IC(θ^{t-1}, θ_L) constrains are indeed satisfied the derived expression for the wages is substituted, which allows me to obtain ($\mathcal{P}_c$).

This approach is similar to that used in Battaglini (2005) and most of the literature of Dynamic Mechanism Design with continuous types. If the agent's post-contractual payoff was a constant or zero, then the solution of ($\mathcal{P}'$) would always be implementable. However, this will no generically be true here because the high type has an additional incentive to pretend to be a low type, as the signal of the latter will generically result in lower posteriors, which is beneficial to her.

Nevertheless, we will show that when the production technology of P_a is sufficiently more efficient from that of P_b then the implementation constrains will be satisfied. However, our sufficient condition for implementation will be relevant only for the following familly of endogenous termination policies.

Definition 1. Call a termination policy *non-decreasing* if for every $t \geq 0$ and realised paths θ^t and $\tilde{\theta}^t$:

$$\theta_{t'} \geq \tilde{\theta}_{t'} \text{ for all } t' \leq t \quad \Rightarrow \quad f_t(\theta^t) \geq f_t(\tilde{\theta}^t)\,.$$

That is a termination policy is non-decreasing if whenever a history θ^t is weakly better than an alternative one $\tilde{\theta}^t$, on each $t' \leq t$, the probability of continuation of the former is higher than that of the latter on all periods up to t. This simple requires that an agent with 'better' history of types is allocated by the contract a higher probability of continuation.

Corollary 3.1 (Point-wise optimal effort)**.** *The point-wise optimal level of effort is*

$$e_t^*(\theta^t) = \begin{cases} (\theta_t)^\epsilon & , \text{ if } \theta^t \neq L^t \\ (\theta_L)^\epsilon/\xi_t & , \text{ if } \theta^t = L^t \end{cases} , \quad \text{where} \quad \xi_t = 1 + \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t \left(1 - \frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \tag{3.3}$$

In addition, if either (i) termination is only exogenous, or (ii) the endogenous termination policy is non-decreasing, then a sufficient condition for the point-wise optimal level of effort to be implementable under any information provision policy is that

$$\frac{\theta_H^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}} - 1\right) - \frac{\theta_L^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(1 - \frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \geq \frac{\delta}{1-\delta}(\rho_H - \rho_L)(1-\kappa)\,\frac{(b\,\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}} \tag{3.4}$$

When $\epsilon = 1$ the above condition simplifies to

$$\frac{1-\kappa}{\kappa} \geq \frac{\delta}{1-\delta}\cdot(\rho_H - \rho_L)\cdot b^2\theta_L^2 \tag{3.5}$$

which is always satisfied for b small enough.

Proof. In Section 7. □

Most of the main results of this specification will be presented for the $\epsilon = 1$ case, which allows us to derive then in closed-form. As a result, the implementation condition that will be used for the rest of the analysis will be (3.5).

3.1.3 Information Provision

This subsection provides some results on the information provision problem of P_a. It will be shown that for the general case it is difficult to write its solution in some ease and meaningful way, however a characterisation of the optimal signal will be provided. More descriptive solution will be derived in the next subsections, where some interesting sub-cases of the model are considered.

P_a's optimal signal will be characterised for any given distribution of the termination time τ, $f = \{f_t\}_{t=0}^{\infty}$, and on each of its possible realisation $t \in \{0, .., \infty\}$. Hence considering only the part of ($\mathcal{P}'$) that is affected by the signal s on realised $\tau = t$ gives

$$\max_{g_t} \left\{ \sum_{\theta^t} \left[\Pr(\theta^t) \Pr(\tau = t \,|\, \theta^t) \sum_s \Pr(\theta_{t+1} = \theta_H | \, \tau = t, \theta_t) B(\beta_t^s) g_t(s \,|\, \theta^t) \right] \right.$$

$$\left. - \, \delta^{t+1} \mu_0 \Pr(\tau = t \,|\, L^t)(\varphi_H - \varphi_L)^t (\rho_H - \rho_L) \sum_s B(\beta_t^s) g_t(s \,|\, L^t) \right\} \qquad (\mathcal{G})$$

The first line of ($\mathcal{G}$) represents the expected information rents from the agent's contract with P_b. Those are captured by P_a through the agent's individual rationality constrains. On the other hand, the second line corresponds to the rents captured by the agent in P_a's contract. Those are due to the fact that whenever the signal s is informative high types have an extra incentive to pretend to be low types, as their continuation payoff $B(\beta)$ is decreasing in their reputation.

Next, ($\mathcal{G}$) is transformed into an equivalent problem that will only take as inputs two posterior beliefs on θ^t, for an agent whose contract was terminated on t. Introduce the following notation

$$\eta_t \;=\; \Pr(\theta_t = \theta_H \,|\, \tau = t), \quad \eta_t^s \;=\; \Pr(\theta_t = \theta_H \,|\, s, \tau = t)$$
$$\lambda_t \;=\; \Pr(\theta^t = L^t \,|\, \tau = t), \quad \lambda_t^s \;=\; \Pr(\theta^t = L^t \,|\, s, \tau = t)$$

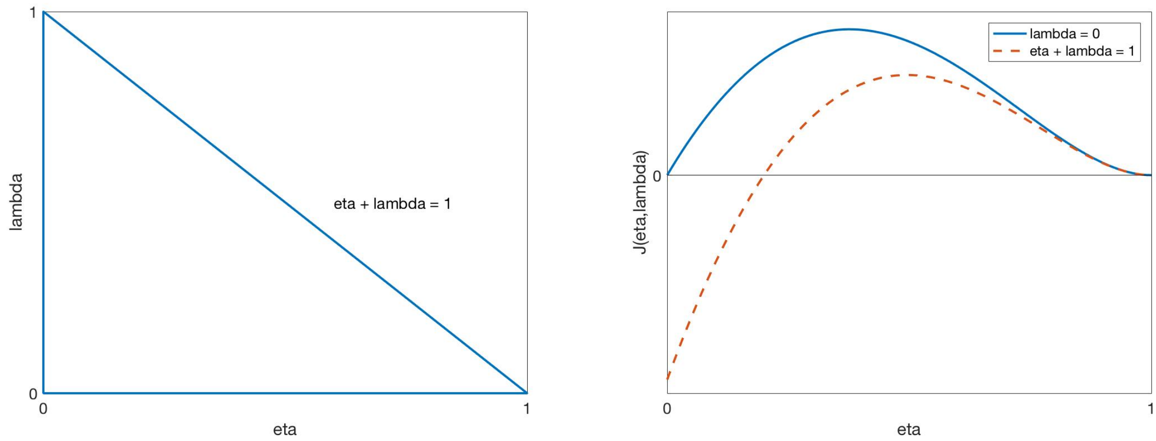
The interim posterior beliefs on the first column only use the information provided by the termination time τ, and those will be the inputs of the equivalent transformation of ($\mathcal{G}$). In contrast, the posteriors on the second column also depend on the signal s; η_t^s is the posterior on $\theta_t = \theta_H$, while λ_t^s on $\theta^t = L^t$. Note that the first event only depends on the contemporaneous θ_t, while the second on the whole history θ^t. Moreover, P_b's posterior on $\theta_{\tau+1}$ and θ_τ are connected according to

$$\beta_t^s \;=\; \rho_H \eta_t^s + \rho_L (1 - \eta_t^s).$$

In the subsequent analysis the underline choice variable will always be the distribution of η_t^s, because this is the one influenced by the signal s, however to facilitate the exposition in many case the results will be presented in terms of β_t^s. Finally, abusing notation let $g_t(s)$ denote the probability of sending signal s after a contract termination at time $\tau = t$, that is

$$g_t(s) \;=\; \sum_{\theta_t} \Pr(\theta^t \,|\, \tau = t) g_t(s \,|\, \theta^t)$$

The rest of the analysis will be based on the following transformation.



(a) Domain (b) Corner values

Figure 2: The domain and the corner values of $J_t(\eta, \lambda)$.

Lemma 3.2. *P_a's information provision problem in period t equivalently becomes*

$$\max_{g_t} \ \mathbb{E}_{g_t}[J_t(\eta_t^s, \lambda_t^s)] \tag{$\mathcal{G}$}$$

where its point-wise value $J_t(\eta_t^s, \lambda_t^s)$ is

$$J_t(\eta_t^s, \lambda_t^s) \ = \ B(\beta_t^s)\big(\beta_t^s - \psi_t \lambda_t^s\big), \quad \textit{and} \quad \psi_t \ = \ \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t (\rho_H - \rho_L)\,. \tag{3.6}$$

Proof. Identical to that of Lemma 1.2 □

Similarly to the previous sections, the information provision problem of P_a is reformulated by considering (η_t^s, λ_t^s) as the underline random variables, instead of the signal s, with joint distribution $\tilde{g}_t$. Hence, P_a equivalently solves

$$\begin{aligned} \max_{\tilde{g}_t} \ \mathbb{E}_{\tilde{g}_t}\big[J_t(\eta, \lambda)\big] \quad \text{s.t. } & \mathbb{E}_{\tilde{g}_t}[\eta] = \eta_t\,, \quad \mathbb{E}_{\tilde{g}_t}[\lambda] = \lambda_t\,, \\ & \eta, \lambda \in [0,1]\,, \text{ and } \eta + \lambda \le 1. \end{aligned} \tag{$\mathcal{G}'_t$}$$

The constrains ensure that the joint distribution $\tilde{g}_t(\eta, \lambda)$ is Bayes plausible.

The domain of $J_t(\eta, \lambda)$ is a right-angled triangle, of which each of its legs has length one. A representative graph of it on the sides where $(\eta + \lambda = 1)$ and $(\lambda = 0)$ is given in plot (2b). Those two sides are connected by straight lines, as $J_t(\eta, \lambda)$ is linear in λ. Note that in plot (2b) the functional form of J_t on both sides is initially increasing and concave, and subsequently changes to decreasing and convex. The next lemma will show that this is a generic representation of those two sides, and will provide a characterisation of $J_t(\eta, \lambda)$ on the rest of its domain. Define functions

$$\begin{aligned} \beta^*(\psi) \ &= \ \frac{1}{2\kappa}\left[2 + \epsilon(1-\kappa) - \sqrt{1-\kappa}\sqrt{4(1+\epsilon) + \epsilon^2(1-\kappa) - \psi 4\kappa(1+\epsilon)}\right] \\ \beta^{**}(\psi) \ &= \ 1 - \frac{\epsilon(1-\kappa)(1-\psi)}{2(1-\kappa\psi) + (1-\kappa)\epsilon} \end{aligned}$$

for generic input ψ. It will be convenient to define $J_t(\eta,\lambda)$ as a function of of η only on the two aforementioned boundaries. Hence, for $i \in \{f,t\}$ let

$$\bar{J}_i(\eta) \;=\; \zeta_i\, B(\beta)(\beta-\Psi_i)\,, \quad \text{where} \quad \begin{cases} \zeta_f = 1 & \Psi_f \;=\; 0 \\ \zeta_t \;=\; 1+\frac{\psi_t}{\rho_H-\rho_L} & \Psi_t \;=\; \frac{\psi_t\rho_H}{\rho_H-\rho_L+\psi_t} \end{cases}$$

and note that $\beta = \eta\rho_H + (1-\eta)\rho_L$ as those two posteriors are always connecting through this linear relationship. Lemma 3.3 shows that $\bar{J}_f(\eta)$ represents the $(\lambda = 0)$ boundary, whereas $\bar{J}_t(\eta)$ the $(\eta+\lambda = 1)$.

Lemma 3.3. *$J_t(\eta,\lambda)$ is neither concave, nor convex on any of the interior points of its domain.*

- *On the boundary $(\eta = 0)$ it is linear and decreasing on λ*
- *On the boundary $(\lambda = 0)$ it satisfies $\bar{J}_f(\eta) = J_t(\eta,0)$*
- *On the boundary $(\eta+\lambda = 1)$ it satisfies $\bar{J}_t(\eta) = J_t(\eta,1-\eta)$*

Moreover, $J_i(\eta)$ is increasing (concave) for

$$\eta \;\le\; \frac{\beta^*(\Psi_i)-\rho_L}{\rho_H-\rho_L} \qquad \left(\eta \;\le\; \frac{\beta^{**}(\Psi_i)-\rho_L}{\rho_H-\rho_L}\right), \tag{3.7}$$

and decreasing (convex) otherwise. Finally, $0 < \beta^*(\Psi_i) \le \beta^{**}(\Psi_i) < 1$.

Proof. In Section 7. □

To solve P_a's information provision problem, first the concave closure of $J_t(\eta,\lambda)$ needs to be characterised. This will be denoted by $\mathcal{J}_t(\eta,\lambda)$ and defined as

$$\mathcal{J}_t(\eta,\lambda) \;=\; \sup\{z\,|\,(\eta,\lambda,z) \in co(J_t)\},$$

where $co(J_t)$ denotes the convex hull of the graph of J_t. In addition, for $i \in \{f,t\}$ let $\bar{\mathcal{J}}_i(\eta)$ denote the value of $\mathcal{J}_t(\eta,\lambda)$ on the boundaries $(\lambda = 0)$ and $(\eta+\lambda = 1)$, respectively, which is also the concave closure of the corresponding $\bar{J}_i(\eta)$. The following proposition characterises $\mathcal{J}_t(\eta,\lambda)$. To facilitate its exposition, a new point needs to be introduced. For $i \in \{f,t\}$ and $\Psi_i < \frac{\theta_L}{\theta_H} < \rho_H$, let $\hat{\eta}_i$ be the unique solution of

$$\bar{J}_i(\hat{\eta}_i) + \bar{J}_i'(\hat{\eta}_i)(1-\hat{\eta}_i) \;=\; 0\,, \tag{3.8}$$

if this exists, and zero otherwise. The functional form of $\hat{\eta}_i$ can be found in the proof of the following proposition in, but it is not copied here, due to its size.

Proposition 3.2. *For any interior point $\mathcal{J}_t(\eta,\lambda) > J_t(\eta,\lambda)$. On the boundary $(\eta = 0)$: $\mathcal{J}_t = J_t$. On the boundaries $(\eta+\lambda = 1)$ and $(\lambda = 0)$:*

- *If $\rho_H \le b^{**}(\Psi_i)$, then $\mathcal{J}_i = J_i$*

- *Otherwise, there exists* $\hat{\eta}_i \in (0,1)$ *such that*

$$\bar{\mathcal{J}}_i(\eta) \;=\; \begin{cases} J_i(\eta) & , \textit{for} \;\; \eta \leq \hat{\eta}_i \\ J_i(\hat{\eta}_i) + J_i'(\hat{\eta}_i)(\eta - \hat{\eta}_i) & , \textit{for} \;\; \eta \geq \hat{\eta}_i \end{cases} . \tag{3.9}$$

where $\hat{\eta}_i = (\hat{\beta}_i - \rho_L)/(\rho_H - \rho_L)$ *and* $\hat{\beta}_i \in \big(\beta^*(\Psi_i), \beta^{**}(\Psi_i)\big)$*. In addition, for* $\epsilon = 1$

$$\hat{\beta}_i \;=\; 1 - \frac{(1-\kappa)^2(\rho_H - \Psi_i)}{2 - (3\rho_H + \Psi_i)\kappa + (\rho_H - \Psi_i + 2\rho_H\Psi_i)\kappa^2}. \tag{3.10}$$

Proof. The statement for the interior points follows from noting that J_t is never concave on any of them. The one for the boundary ($\eta = 0$) follows because the function is linear with respect to λ.

Finally, for the boundaries ($\lambda = 0$) and ($\eta + \lambda = 1$) note that if $\rho_H \leq \beta^{**}(\Psi_i)$, then $J_i(\eta)$ does not have a convex part. Otherwise, it is convex on $\eta > \frac{\beta^{**}(\Psi_i) - \rho_L}{\rho_H - \rho_L}$. Hence, there is a unique line that connects $\eta = 1$ with some $\hat{\eta}_i$ between the maximum of J_i and the point on which its concavity changes. This is defined as the solution of

$$J_i(1) \;=\; J_i(\hat{\eta}_i) + J_i'(\hat{\eta}_i)(1 - \hat{\eta}_i)\,. \tag{3.11}$$

For $\epsilon = 1$, $\hat{\eta}_i$ can be given in closed form. To solve this let $\widetilde{J}_i(\beta) \;=\; J_i[(\beta - \rho_L)/(\rho_H - \rho_L)]$, that is define a function that has β as the underline variable instead of η. Then the graph of $\tilde{J}_i(\beta)$ on $[\rho_L, \rho_H]$ is equal to that of $J_i(\eta)$ on $[0,1]$. Hence, solve

$$\widetilde{J}_i(\rho_H) \;=\; \widetilde{J}_i(\hat{\beta}_i) + \widetilde{J}_i'(\hat{\beta}_i)(\rho_H - \hat{\beta}_i)\,, \tag{3.12}$$

which can be solved to obtain (3.10). □

Therefore, we are ready to state our first result.

Corollary 3.2. *Suppose that* (η_t, λ_t) *are interior points, then there is always information provision. On the boundary* ($\eta = 0$) *no information provision is optimal. On the boundaries* ($\lambda = 0$) *and* ($\eta + \lambda = 1$)

- *If* $\rho_H \leq b^{**}(\Psi_i)$*, no information provision is optimal*
- *Otherwise if* $\eta_t > e\hat{t}a_i$*, then any optimal signal must randomise between* $\hat{\eta}_i$ *and 1.*

3.2 Exogenous Termination

This section considers two distinct specifications. The first one solves the information provision problem of P_a under the assumption that $\gamma = 0$ and finds a simple sufficient condition for implementation. Imposing $\gamma = 0$ restricts the probability of the contract to continue after period 0 to zero. This is equivalent to considering a situation where P_a is exogenously restricted to offer contracts of fix time length, for example when the task to be completed by the agent is not recurring.

The second specification that will be considered will assume that $\gamma > 0$, but will restrict P_a to offering tenure contracts. That is in this case the contract will only be terminated because of the exogenous τ_γ. For the sake of the exposition assume that $\epsilon = 1$, for both of the above specifications.

3.2.1 Fix term contracts

Here we solve ($\mathcal{G}_t'$) for the case where $\gamma = 0$. In particular, we only need to solve it for $t = 0$, since the contract will always be terminated at the end of period 0. Thus, it has to be that $\eta + \lambda = 1$, because $\theta_0 = \theta_H$ and $\theta_0 = \theta_L$ are the only two possible histories in period 0. Hence

$$J_0(\eta, \lambda) \;=\; J_0(\eta, 1-\eta) \;=\; \bar{J}_0(\eta)$$

Hence hereafter we will simple write $J_0(\eta)$. In addition, some algebra gives that $\Psi_0 = \mu_0 \rho_H$. Hence, substituting in the expression of $\bar{J}_0(\eta)$ gives that

$$J_0(\eta) \;=\; B(\beta) \cdot \frac{\beta - \mu_0 \rho_H}{1 - \mu_0}$$

As a result, ($\mathcal{G}_t'$) reduces to

$$\max_{\tilde{g}_0} \; \mathbb{E}_{\tilde{g}_0}\big[J_0(\eta)\big] \quad \text{s.t.} \;\; \mathbb{E}_{\tilde{g}_0}[\eta] = \mu_0 \,, \quad \eta \in [0,1]. \tag{3.13}$$

In particular, the optimal signal needs to satisfy $\mathbb{E}_{\tilde{g}_0}\big[J_0(\eta)\big] = \mathcal{J}_0(\mu_0)$, where $\mathcal{J}_0$ is the concave closure of J_0. Hence the concave closure of $J_0(\eta)$ needs to be characterised, in order to solve (3.13). However, this has already be done in Proposition 3.2. Let $\beta_0^{**} = \beta^{**}(\mu_0 \rho_H)$ and remember that we have shown that if $\rho_H \leq \beta_0^{**}$, then $\mathcal{J}_0 = J_0$. Otherwise, there exists $\hat{\eta}_0 = (\hat{\beta}_0 - \rho_L)/(\rho_H - \rho_L)$ such that

$$\mathcal{J}_0(\eta) \;=\; \begin{cases} J_0(\eta) & \text{, for } \;\eta \leq \hat{\eta}_0 \\ J_0(\hat{\eta}_i) + J_0'(\hat{\eta}_0)(\eta - \hat{\eta}_0) & \text{, for } \;\eta \geq \hat{\eta}_0 \end{cases} , \tag{3.14}$$

$$\hat{\beta}_0 \;=\; 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2 - (3+\mu_0)\rho_H\kappa + (1 - \mu_0 + 2\mu_0\rho_H)\rho_H\kappa^2}. \tag{3.15}$$

P_b's posterior on θ_0 is not affected by the termination time, because this is not correlated with the agent's type. Hence, the optimal signal structure has to satisfy $\mathbb{E}_{\tilde{g}_0}[J_0(\eta)] = \mathcal{J}_0(\mu_0)$, which implies the following necessary and sufficient condition for information provision to be optimal in (3.13)

$$\rho_H \;\geq\; \beta_0^{**} \quad \text{and} \quad \mu_0(\rho_H - \rho_L) + \rho_L \;\geq\; \hat{\beta}_0 \tag{3.16}$$

The first inequality ensures that the convex part of $J_0(\eta)$ exists, and the second that μ_0 is big enough for the prior on θ_1 to be on this convex part.

Proposition 3.3 (Fixed Term Contracts)**.** *An informative signal strictly solves* (3.13) *if and only if*

$$\begin{aligned} \rho_H \;&>\; 1 - \frac{(1-\kappa)(1-\mu_0\rho_H)}{2(1-\kappa\mu_0\rho_H) + (1-\kappa)} \qquad \textit{and} \\ \mu_0(\rho_H - \rho_L) + \rho_L \;&\geq\; 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2 - (3+\mu_0)\rho_H\kappa + (1-\mu_0+2\mu_0\rho_H)\rho_H\kappa^2} \end{aligned} \tag{3.17}$$

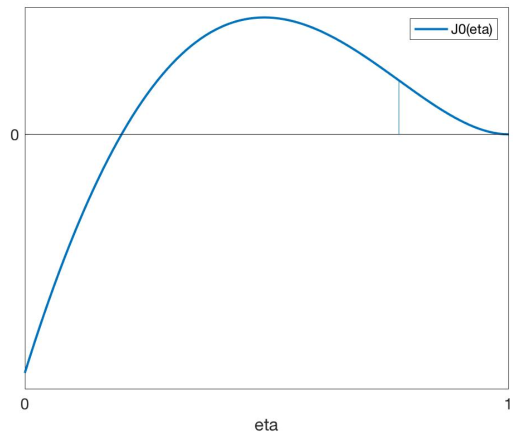


Figure 3: $J_0(\eta)$

The higher ρ_H is the less binding the first line of (3.17) becomes. Also, suppose that the first line is satisfied, then the higher either ρ_H, or ρ_L is, the less the second line of (3.17) binds. When this is satisfied the optimal signal $s \in \{s_L, s_H\}$ has distribution

$$g_0(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_0(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_0}{\mu_0}\frac{\hat{\eta}_0}{1-\hat{\eta}_0}. \tag{3.18}$$

Proof of Proposition 3.3. Condition (3.17) follows by substituting the functional forms of β_0^{**} and $\hat{\beta}_0$ in (3.16). Next, the statements on its dependence on ρ_L and ρ_H is proven. For the first line re-write inequality as

$$\rho_H[2(1-\kappa\mu_0\rho_H)+(1-\kappa)] \;\geq\; 2(1-\kappa\mu_0\rho_H)+(1-\kappa)-(1-\kappa)(1-\mu_0\rho_H).$$

Differentiate both sides with respect to ρ_H, and subtract the derivative of the right hand side from that of the left one to obtain

$$\begin{aligned} 2(1-\kappa\mu_0\rho_H)+(1-\kappa)-2\kappa\mu_0\rho_H+2\kappa\mu_0-\mu_0(1-\kappa) \;&=\; \\ 2(1-\kappa\mu_0\rho_H)+(1-\mu_0)(1-\kappa)+2\kappa\mu_0(1-\rho_H) \;&\geq\; 0. \end{aligned}$$

As a result the more ρ_H increases the less binding this inequality becomes. To prove the same for the second line calculate

$$\frac{\partial\hat{\beta}_0}{\partial\rho_H} \;=\; \frac{-(1-\kappa)^2(1-\mu_0)2(1-\mu_0\rho_H^2\kappa^2)}{2-(3+\mu_0)\rho_H\kappa+(1-\mu_0+2\mu_0\rho_H)\rho_H\kappa^2}$$

which is negative for $\rho_H > \beta_0^{**}$. This is because in this case $\hat{\beta}_0 < \beta_0^{**} < 1$, which implies that the denominator above have to be positive. Hence if $\rho_H > \hat{\beta}_0^{**}$, as ρ_H increases the left hand side increasing and the right hand side decreases.

Finally, the optimal signal is obtained by the following argumentation. Suppose that $\rho_H \leq \beta_0^{**}$, then $J_0(\eta)$ is concave for all $\eta \in [0,1]$, hence an uninformative signal is optimal. Instead suppose that $\rho_H > \beta_0^{**}$, then the convex hull of J in linear on $[\hat{\eta}_0, 1]$ and strictly concave everywhere else. Hence if $\mu_0 \le \hat{\eta}_0$, then no information provision is still optimal. If $\mu_0 > \hat{\eta}_0$ then the optimal signal randomises between posteriors $\hat{\eta}_0$ and 1. Let s_H be the signal that fully reveals the high type. Then the probability of sending s_L is obtained by solving

$$\hat{\eta}_0 \;=\; \frac{\mu_0 g_0(s_L \,|\, \theta_H)}{\mu_0 g_0(s_L|\theta_H) + 1 - \mu_0}. \tag{3.19}$$

□

To understand this result note that $B(\beta)$, which represents the agent's information rents from his contract with P_b, is convex[7]. This provides an incentive towards information provision for P_a, because she captures the agent's continuation value through his individual rationality constrain. However, she also has to pay information rents to the high type that are proportional to his continuation value, which create an incentive towards the opposite direction. For high μ_0 the information rents that P_a pays are low enough for the first incentive to dominate, while the opposite is true for low μ_0.

The optimal signal, under information provision, has a realisation s_H that reveals the high type, and one s_L that is always sent for the low type, but also some times for the high type. However, the proposed optimal signal may not always be implementable, as under the point-wise optimal level of effort the implementation constrain (7.4) needs to hold. However, this is implied by (3.5), hence our solution is relevant for at least a subset of parameters.

A case of special interest is the following one.

Example 3.1 (Privacy)**.** Suppose that $\rho_L = 0$ and $\rho_H = 1$. Then no information provision is optimal and implementable.

Proof. For $(\rho_L, \rho_H) = (0, 1)$ the second line of (3.17), which is a necessary condition for information provision, is satisfied, while the first becomes

$$\mu_0 \;\ge\; 1 - \frac{(1-\kappa)^2(1-\mu_0)}{2-(3+\mu_0)\kappa + (1-\mu_0+2\mu_0)\kappa^2} \;\Leftrightarrow\; \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2} \;\ge\; 1$$

If the denominator is negative, then this cannot hold. If it is positive, then

$$\frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2} \;<\; \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa} \;=\; \frac{(1-\kappa)^2}{2(1-\kappa)} \;=\; \frac{1-\kappa}{2} \;<1,$$

hence again the above inequality cannot hold. □

Proposition 3.3 gives that the necessary and sufficient condition for information provision (3.17) binds less as ρ_H increases. Hence, if privacy is optimal for $(\rho_L, \rho_H) = (0, 1)$, then the same is true for any $\rho_H < 1$ and $\rho_L = 0$. Hence, in order to break the no information provision result, as identified in Calzolari and Pavan (2006)[8], it has to be that an initially low type has at least some positive probability of being a high type under P_b.

[7]Even though depending on the ϵ this may not always be true for all its domain, it is still a generic result that $B(b)$ is convex for high b, which suffices to obtain the kind of results provided above.

[8]The reason why the result does not hold in this setting is because the agent's type is not perfectly correlated across employments. The cited paper identifies a few other reasons.

Example 3.2 (Information Provision)**.** Assume that $\rho_L > 1/2$ and $\kappa \to 0$, then information provision is optimal and implementable.

Proof. As in the proof of the previous example $\rho_H = 1$ gives that the second line of (3.17) is satisfied, while the first becomes

$$\rho_L \ \geq \ 1 - \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2},$$

the right hand side of which goes to $1/2$ as $\kappa \to 0$. Moreover, it has already being argued, using (3.5), that for $\kappa \to 0$ information provision is implementable. □

3.2.2 Tenure Contracts

Skill Accomolation

The first part of this subsection assumes that $\varphi_H = 1$, that is it assumes that the agent can only become better as time passes. Define recursively the posterior belief of a type that is employed In period t as

$$\mu_t \ = \ \mu_{t-1}(\varphi_H - \varphi_L) + \varphi_L \quad \Rightarrow \quad \mu_t \ = \ \mu_0(\varphi_H - \varphi_L)^t + \varphi_L \, \frac{1-(\varphi_H-\varphi_L)^t}{1-(\varphi_H-\varphi_L)}$$

and note that $\mu_{t+1} > \mu_t$. As the termination time is not correlated with the agent's type, the posterior reputation of an agent that is terminated on $t+1$ is $\eta_t = \mu_t$. To solve P_a information provision problem note that for an agent that is terminated on $t+1$ there are only two possible events, either $\theta_t = \theta_H$, or $\theta^t = L^t$. As a result it has to be that $\eta + \lambda = 1$. Hence, similarly to before

$$J_0(\eta, \lambda) \ = \ J_0(\eta, 1-\eta) \ = \ \bar{J}_0(\eta)$$

In addition, algebra identical to that of the previous section shows that $\varphi_H = 1$ implies $\Psi_t = \mu_0 \rho_H$. Hence, substituting in the expression of $\bar{J}_t(\eta)$ gives that

$$J_t(\eta) \ = \ J_0(\eta) \ = \ B(b) \cdot \frac{b - \mu_0 \rho_H}{1-\mu_0}$$

Hence, P_a information provision problem becomes

$$\max_{\tilde{g}_t} \ \mathbb{E}_{\tilde{g}_t}\big[J_0(\eta)\big] \quad \text{s.t.} \ \ \mathbb{E}[\eta] = \mu_t \, , \quad \eta \in [0,1]. \tag{3.20}$$

In particular, the optimal signal needs to satisfy $\mathbb{E}_{\tilde{g}_t}\big[J_0(\eta)\big] = \mathcal{J}_0(\mu_t)$. This is identical to P_a's problem under fixed term contracts, however now instead of μ_0, the restriction on the distribution of posteriors is that $\mathbb{E}_{\tilde{g}_t}[\eta] = \mu_t$. Hence, even if μ_0 is such that no information provision is optimal In period 0, it is still possible for this result to be reversed as time progresses.

Proposition 3.4 (Skill Accomulation)**.** *An informative signal strictly solves* (3.20) *if and only if*

$$\begin{aligned} \mu_t(\rho_H - \rho_L) + \rho_L \;&\geq\; 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2-(3+\mu_0)\rho_H\kappa + (1-\mu_0+2\mu_0\rho_H)\rho_H\kappa^2} \\ \textit{and} \qquad \rho_H \;&>\; 1 - \frac{\epsilon(1-\kappa)(1-\psi)}{2(1-\kappa\psi)+(1-\kappa)\epsilon} \end{aligned} \tag{3.21}$$

The higher ρ_H *is the less binding the second line of* (3.21) *becomes. Also, suppose that the second line is satisfied, then the higher either* ρ_H*,* ρ_L*, or* μ_t *is, the less the first line of* (3.21) *binds. When this is satisfied the optimal signal* $s \in \{s_L, s_H\}$ *has distribution*

$$g_t(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_t(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_t}{\mu_t}\frac{\hat{\eta}_0}{1-\hat{\eta}_0}. \tag{3.22}$$

Proof. The second line of (3.21) and the right hand side of its first line follows from the functional form of J_0, hence they are not affected by the evolution of beliefs over time. In contrast, the passage of time affects the left hand side of the first line, which represent the prior belief of a type that is terminated on $t+1$. Similarly, $\hat{\eta}_0$ is only a function of the functional form of J_0 so it is not affected by μ_t. □

Long Contracts

Next drop the assumption that $\varphi_H = 1$. That is assume that $0 \leq \varphi_L \leq \varphi_H < 1$, but only consider the information provision problem of P_a as $t \to \infty$. In this case P_b's posterior on an buyer whose contract has not been terminated in period t is

$$\lim_{t\to\infty} \mu_t \;=\; \mu_\infty \;=\; \frac{\varphi_L}{1-(\varphi_H - \varphi_L)}\,,$$

To state the condition for information provision on the limit, note that Proposition 3.2 has already defined $\hat{\eta}_f = \frac{\hat{\beta}_f - \rho_L}{\rho_H - \rho_L}$ and

$$\hat{\beta}_f \;=\; 1 - \frac{(1-\kappa)^2\rho_H}{2-3\rho_H\kappa+\rho_H\kappa^2}\,.$$

Proposition 3.5 (Long Contracts)**.** *Information provision is strictly optimal on the steady state of* ($\mathcal{G}_t'$)*, under tenure contracts, if and only if*

$$\begin{aligned} \mu_\infty(\rho_H - \rho_L) + \rho_L \;&\geq\; 1 - \frac{(1-\kappa)^2\rho_H}{2-3\rho_H\kappa+\rho_H\kappa^2} \\ \textit{and} \qquad \rho_H \;&>\; \frac{2}{3-\kappa} \end{aligned} \tag{3.23}$$

The higher ρ_H *is the less binding the second line of* (3.23) *becomes. Also, suppose that the second line is satisfied, then the higher either* ρ_H*,* ρ_L*, or* μ_∞ *is, the less the first line of* (3.23) *binds. When this is satisfied the optimal signal* $s \in \{s_L, s_H\}$ *has distribution*

$$g_\infty(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_\infty(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_\infty}{\mu_\infty}\frac{\hat{\eta}_\infty}{1-\hat{\eta}_\infty}. \tag{3.24}$$

Proof. As already stated J_∞ is the limit of J_t for $t \to \infty$. Hence, the relevant condition for information provision can be obtained by substituting the points already derived in the previous section. In particular, the necessary and sufficient condition for J_∞ to have a convex part is $\rho_H > \beta^{**}(0)$, which gives the second line of (3.23). When this is satisfied for the prior to be big enough to be on the linear part it has to be that $\mu_\infty(\rho_H - \rho_L) + \rho_L \geq \hat{\beta}_f$, where the functional form of $\hat{\beta}_f$ is given in Proposition 3.2.

All of the statement regarding (3.23) and when it is more binding follow from simple differentiation. The statement on the optimal signal under information provision follows noting that this has to randomise between $\hat{\eta}_f$ and one, while the expected value of the posterior needs to remain μ_∞. □

Finally, note that because $\gamma > 0$ is a non-decreasing termination policy it follows that (3.5) is a sufficient condition for implementation. Moreover, this always holds for $\kappa \to 0$, which turns (3.23) into

$$\mu_\infty(\rho_H - \rho_L) + \rho_L \;\geq\; 1 - \frac{\rho_H}{2} \quad \text{and} \quad \rho_H \;>\; \frac{2}{3} \tag{3.25}$$

Example 3.3 (Information Provision)**.** Let $\kappa \to 0$, $\rho_H \to 1$, then information provision is optimal on the steady state, and any solution is implementable on all the path, if and only if

$$\mu_\infty(1 - \rho_L) + \rho_L \;\geq\; \frac{1}{2},$$

which is satisfied for $\rho_L \geq 1/2$, and not satisfied for $\rho_L = \varphi_L \to 0$.

4 Endogenous termination

The model cosidered here is that introduced in Section 3. Here, we allow P_a to commit in advance on some probability of terminating a contract after the realisation of a certain history of reported types. To make the results more tractable assume throughout this section that $\varphi_H = 1$, which as argued before in the beginning of the subsection on tenure contracts implies that $\mathcal{J}_t(\eta, \lambda) = J_0(\eta)$. Moreover, $\xi_t = \xi_0$. To maintain the notation as light as possible, let

$$u_H \;=\; \frac{\theta_H^{1+\epsilon}}{1+\epsilon} \quad \text{and} \quad u_l \;=\; \frac{\theta_L^{1+\epsilon}}{\xi_0^\epsilon(1+\epsilon)}\,, \tag{4.1}$$

represent the point-wise optimal flow payoffs of P_a from a high and low type, respectively. Note that once a low type turns high it remains so, hence those two are the only relevant possibilities in terms of flow payoffs. Moreover, let

$$f_t \;=\; \Pr(\tau_a > t+1 \,|\, \tau_a > t) \quad \text{and} \quad x_t(\theta^t) \;=\; \Pr(\tau_a > t+1 \,|\, \tau_a > t, \theta^t) \tag{4.2}$$

As a result, once the point-wise optimal effort and signal have being substituted in ($\mathcal{P}'$), this obtain the following recursive representation

$$V_t(\mu_t) \;=\; \max_{x(\theta^t)\in[0,1]} \; \mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t), \tag{4.3}$$

where note that the past history of a high type does not matter, while there is only one history on which the agent is a low type in t, that is L^t. Hence, it is without loss to condition $x_t(\theta^t)$ only on the current type and time, hence write x_t^H and x_t^l and note that

$$f_t \;=\; \mu_t x_t^H + (1-\mu_t)x_t^l\,, \qquad \eta_t \;=\; \frac{\mu_t(1-\gamma x_t^H)}{1-\gamma f_t}\,, \qquad \mu_{t+1} \;=\; \frac{\mu_t x_t^H + (1-\mu_t)x_t^l\varphi_L}{f_t}\,.$$

The treatment of the above problem is quite tedious and can be found in Section 8 . To reduce the number of case that need to be considered the following assumption is imposed.

Assumption 1. $\mathcal{J}_0$ is twice continuously differentiable and concave. Both u_H and u_l are positive. Also,

$$u_H \;>\; (1-\delta)\mathcal{J}_0(1) \qquad \text{and} \qquad \frac{u_H}{1-\delta\gamma} + \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1) \;>\; \mathcal{J}_0(0) + \mathcal{J}_0'(0). \tag{4.4}$$

The main result which characterises the solution of P_a optimal stopping problem is given below.

Proposition 4.1. *Continuing a high type, with probability one, is always strictly optimal. Stopping a low type, with probability one, for every $\mu_t \in [0,1]$ is strictly suboptimal if*

$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1) \;>\; 0 \tag{4.5}$$

is satisfied. In contrast, if it holds in the reversed direction and

$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l + \delta\left[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma} + \gamma(1-\varphi_L)\right]\mathcal{J}_0(1) - \delta(1-\varphi_L)\mathcal{J}_0'(1) \;<\; \mathcal{J}_0(0) \tag{4.6}$$

is satisfied, then stopping a low type, with probability one, is optimal for all $\mu_t \in [0,1]$. Otherwise, there exists $\tilde{\mu}$ such that stopping a low type, with probability one, is optimal if and only if $\mu_t > \tilde{\mu}$.

Whenever the low type is not stopped, with probability one, the reputation of both a terminated η_t and non-terminated μ_{t+1} agent increases over time.

Example 4.1. Let $\rho_L = \mu_0\rho_H$ and $\rho_H = 1$. Then for u_L and φ_L small enough the optimal contract has the low type to be fired with probability one, one for some interior $\tilde{\mu}$.

Proof. Under the above choice of parameters $\mathcal{J}_0(0) = \mathcal{J}_0(1) = 0$ and $\mathcal{J}_0'(1) < 0$. As a result neither of the above two inequalities hold when

$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l \;<\; -\mathcal{J}_0'(1)[1-\delta(1-\varphi_L)]$$
$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l \;>\; \delta(1-\varphi_L)\mathcal{J}_0'(1)$$

The second inequality always hold. The first inequality holds if the left hand side is small enough, which is true when φ_L and u_l and small enough. ∎

5 Proofs for multi-period contracts

Proof of Lemma 1.1. The dependence on t and s is dropped. The revelation principle applies. To make the notation more compact, write the reported type as a subscript. S_2's revenue maximisation problem is the following one

$$\begin{aligned}
\max_{p,q} \quad & \beta \cdot \Big(p_H - c(q_H)\Big) + (1-\beta) \cdot \Big(p_L - c(q_L)\Big) \\
\text{s.t. } (\mathrm{IR}_L) \quad & b\,\theta_L\, q_L - p_L \;\geq\; 0 \\
(\mathrm{IR}_H) \quad & b\,\theta_H q_H - p_H \;\geq\; 0 \\
(\mathrm{IC}_L) \quad & b\,\theta_L\, q_L - p_L \;\geq\; b\,\theta_L\, q_H - p_H \\
(\mathrm{IC}_H) \quad & b\,\theta_H q_H - p_H \;\geq\; b\,\theta_H q_L - p_L
\end{aligned}$$

where both the constrains and the objective function are written in terms of per period payoffs. Assuming that (IR_L) does not bind leads to a contradiction. Subsequently, this can be used to show that (IC_H) has to bind. Hence the above simplifies to the unconstrained maximisation problem

$$\max_q \; \beta \cdot \Big(b\,\theta_H\, q_H - c(q_H)\Big) + (1-\beta) \cdot \Big(b\,\frac{\theta_L - \beta\theta_H}{1-\beta} q_L - c(q_L)\Big)$$

For $\beta < \theta_L/\theta_H$ the objective function is concave, hence the unique solution is given by the first order conditions

$$c'(q_H) \;=\; b \cdot \theta_H \quad \text{and} \quad c'(q_L) \;=\; b \cdot \frac{\theta_L - \beta\theta_H}{1-\beta}$$

This is implementable, because substituting the above solutions in (IC_L) gives

$$\begin{aligned}
b\,\theta_L q_L - p_L \;\geq\; b\,\theta_L q_H - p_H \;\Leftrightarrow\; 0 \;\geq\; b\,\theta_L q_H - p_H \\
\Leftrightarrow\; b\,\theta_H(q_H - q_L) + b\,\theta_L q_L \;\geq\; b\,\theta_L q_H \;\Leftrightarrow\; q_H \;\geq\; q_L,
\end{aligned}$$

which is satisfied. Because the (IR_L) binds the low type's period payoff is zero. The high type's period payoff can be obtained using the (IR_L) and (IC_H) constrains, which give that

$$b\,\theta_H q_H - p_H \;=\; b\,\theta_H q_L - p_L \;=\; b\,(\theta_H - \theta_L) q_L \;=\; b\,(\theta_H - \theta_L)\left(b\,\frac{\theta_L - \beta\theta_H}{1-\beta}\right)^{\epsilon}$$

Hence a constant stream of the above payoff up to infinity gives $B(\beta)$. To obtain the results on its derivatives note that $\frac{\theta_L - \beta\theta_H}{1-\beta} = \theta_H - \frac{\theta_H - \theta_L}{1-\beta}$. Hence, on its non-flat part

$$B'(\beta) \;=\; -b^{1+\epsilon}\,(\theta_H - \theta_L)\left(\frac{\theta_L - \beta\theta_H}{1-\beta}\right)^{\epsilon-1} \epsilon\,\frac{\theta_H - \theta_L}{(1-\beta)^2}$$

Then the first expression below is obtained by gathering terms, while the second from differentiating again.

$$B'(\beta) \;=\; -\frac{B(\beta)}{1-\beta}\,\epsilon\,\frac{\theta_H - \theta_L}{\theta_L - \beta\theta_H} \quad \text{and} \quad B''(\beta) \;=\; \frac{B'(\beta)}{1-\beta}\left(2 + (1-\epsilon)\frac{\theta_H - \theta_L}{\theta_L - \beta\theta_H}\right), \tag{5.1}$$

Thus, the statements on the monotonicity and concavity of $B(\beta)$ on its non-flat part follow immediately from the above. □

Lemma 5.1. *It is without loss of generality to only consider one-shot deviations. In those a θ^t buyer type reports truthfully θ^{t-1}, potentially misreports θ_t as $\hat{\theta}_t$, and subsequently switches back to truthful reporting.*

Proof of Lemma 5.1. Necessity is trivial. For sufficiency suppose that type θ^t has a profitable deviation to report $\{\hat{\theta}_t, .., \hat{\theta}_{t'}\}$ up to t' and then switch to truthfulness. But on t' a realised type $\{.., \hat{\theta}_t, .., \hat{\theta}_{t'-1}, \theta_{t'}\}$ that was truthful faces the same payoff on t' as a misreported one that ends with type $\theta_{t'}$. Hence, the one-shot deviation constrains implies that $\{\hat{\theta}_t, .., \hat{\theta}_{t'-1}, \theta_{t'}\}$, that is a deviation that misreports only up to $t' - 1$ is better. Applying the same argument shows that any deviation with finite horizon t' is no better than truth-telling. Infinitely long deviations can be arbitrarily well approximated by finite ones, hence as long as the one-shot constrains bind up to a constant difference $\varepsilon > 0$ the same contradiction is obtained. To maintain notation light this ε is ignored on the main text. □

Proof of Proposition 1.1. Lemma 5.1 shows that it is without loss to only consider one shot deviations. Hence hereafter IC(θ^t) will refer exclusivity to the incentive compatibility constrains obtained under one-shot deviations. To maintain a compact notation, let $\hat{\theta}^t = \{\theta^{t-1}, \hat{\theta}_t\}$ denote a history of truthful reports up to $t-1$ followed by a potential misreport $\hat{\theta}_t$. In addition, denote a generic history θ^{t-1} followed by $\theta_t = \theta_H$ as θ^t_H, and similarly define θ^t_L. Then the payoff of a θ^t buyer type under a one shot deviation is

$$\begin{aligned}\widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1}) \;=\; & \theta_t q_t(\hat{\theta}^t) - p_t(\hat{\theta}^t) + \gamma\delta\, \mathbb{E}_\theta\big[\widehat{U}_{t+1}(\theta_{t+1}, \theta_{t+1}, \hat{\theta}^t) \,|\, \theta_t\big] \\ & + (1-\gamma)\,\delta\, \Pr(\theta_{t+1} = \theta_H \,|\, \tau = t, \theta_t)\, \mathbb{E}_g\big[B(\beta^s_t) \,|\, \hat{\theta}^t\big]\end{aligned}$$

In period t the buyer obtains the quantity and price corresponding to type $\hat{\theta}_t$, however his actual valuation corresponds to the realised type θ_t. The distribution of the signal s is conditioned on the reported type $\hat{\theta}^t$, but the probability of the buyer to be a high type in S_2's contract is only a function of the actual type θ_t. Let the on path payoff of a θ^t buyer type be given by $U_t(\theta^t) = \widehat{U}_t(\theta_t, \theta_t, \theta^{t-1})$, then the corresponding individual rationality and incentive compatibility constrains become

$$\begin{aligned}\text{IR}(\theta^t) \quad & U_t(\theta^t) \;\geq\; 0 \\ \text{IC}(\theta^t) \quad & U_t(\theta^t) \;\geq\; \widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1})\end{aligned} \tag{5.2}$$

where the buyer's outside option is set to zero, since we have assumed that if the buyer rejects S_1's offer, then S_2 does not trade with him. Next, consider the following problem

$$(\mathcal{P}^H) \quad \max_{q,g}\; \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \delta^t \Big(p_t(\theta^t) - c\big[q_t(\theta^t)\big]\Big)\right] \quad \text{subject to IR}(\theta^t) \text{ and IC}(\theta^t), \tag{5.3}$$

which is similar to ($\mathcal{P}$), but it ignores the individual rationality constrains of all high types IR(θ^t_H) and the incentive compatibility constrains of all low types IC(θ^t_L). Ignoring the former set of constrains is without loss of generality as $U_t(\theta^t_H) \geq U_t(\theta^t_L)$, however this is not true for the IC(θ^t_L) constrains. Despite that, if the solution of ($\mathcal{P}^H$) happens to also satisfy this set of constrains, then it is a solution of ($\mathcal{P}$).

The rest of the proof shows that in $(\mathcal{P}^H)$, for every (p,q,g) there exists a p' such that the objective function under (p',q,g) is no less than under (p,q,g), and that both the IR(θ_L^t) and IC(θ_H^t) constrains bind. Solving for p' from the constrains and substituting in the objective function will give $(\mathcal{P}')$. Hence if (q,g) is a solution of $(\mathcal{P}')$, then there exists p' such that (p',q,g) is also a solution of $(\mathcal{P}^H)$. Finally, p' will be substituted in the previously ignored IC(θ_L^t) so that a sufficient condition is obtained for (p',q,g) to be a solution of $(\mathcal{P})$, which only depends on policies (q,g).

The argument is recursive. Suppose that IR(θ_L^t) and IC(θ_H^t) bind for all periods up to and including t', but not for $t'+1$. For simplicity, denote $p_t(\theta_H^{t'})$ by p_H, $p_t(\theta_L^{t'})$ by p_L, and similarly use $\{p_{LL}, p_{HL}, p_{LH}, p_{HH}\}$ for the possible combinations up to $t'+1$. Moreover, adopt the same notational change for the IR and IC constrains. Suppose that IR$_{HL}$ does not bind, then let

$$\big(\tilde{p}_H, \tilde{p}_{HH}, \tilde{p}_{HL}\big) \;=\; \big(p_H - \delta\varepsilon,\, p_{HH}+\varepsilon,\, p_{HL}+\varepsilon\big)$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is indifferent between the two contracts. The same argument works if IR$_{LL}$ does not bind. Suppose instead that IC$_{LH}$ does not bind, then let

$$\big(\tilde{p}_L,\, \tilde{p}_H,\, \tilde{p}_{LH}\big) \;=\; \big(p_L - \delta\varphi_L\varepsilon,\, p_H + \delta(\varphi_H-\varphi_L)\varepsilon,\, p_{LH}+\varepsilon\big),$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is actually better off. Finally, suppose that IC$_{HH}$ does not bind, then let

$$\big(\tilde{p}_H,\, \tilde{p}_{HH}\big) \;=\; \big(p_H - \delta\varphi_H\varepsilon,\, p_{HH}+\varepsilon\big),$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is indifferent between the two contracts. Hence, if both IR(θ_L^t) and IC(θ_H^t) bind for all periods up to and including t', and IC(θ_L^t) is ignored, then there exists an alternative contract that implements the same policies, is not worse for S$_1$, and has all constrains binding up to $t'+1$. In addition, the regular one period argumentation shows that IR(θ_L^0) and IC(θ_H^0) have to bind, from which the recursive argument follows.

Hence, it is without loss to assume that IR(θ_L^t) and IC(θ_H^t) bind. The former gives

$$p_t(\theta_L^t) \;=\; \theta_L q_t(\theta_L^t) + \gamma\delta\varphi_L \widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + (1-\gamma)\delta\rho_L \mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]\,,$$

and the latter

$$\begin{aligned} U_t(\theta_H^t) \;=\; &\theta_H q_t(\theta_L^t) - p_t(\theta_L^t) + \gamma\delta\varphi_H \widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) \\ &+ [1-\gamma]\delta\rho_H \mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]\,. \end{aligned}$$

In both equations above it has been used that $\widehat{U}_{t+1}(\theta_L,\theta_L,\theta_L^t) = U_{t+1}(\{\theta_L,\theta_L^t\})$, which in turn is equal to zero because the IR$(\{\theta_L,\theta_L^t\})$ constrain binds. Substitute the derived expression for $p_t(\theta_L^t)$ in that for $U_t(\theta_H^t)$ to obtain

$$\begin{aligned} U_t(\theta_H^t) \;=\; &(\theta_H-\theta_L) q_t(\theta_L^t) + \gamma\delta(\varphi_H-\varphi_L) \widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) \\ &+ (1-\gamma)\delta(\rho_H-\rho_L) \mathbb{E}_g[B(\beta_t^s)|\theta_L^t]\,. \end{aligned} \tag{5.4}$$

In addition, because $\text{IC}(\{\theta_L^t, \theta_H\})$ binds, I get that $\widehat{U}_{t+1}(\theta_H, \theta_H, \theta_L^t) = U_{t+1}(\{\theta_L^t, \theta_H\})$, the expression of which follows the same pattern with the above equation. Using the same argument repeatedly and substituting forward gives the functional form provided for $U_t^H(\theta^{t-1})$ in the main text. In particular, for a period 1 high types that becomes

$$U_0(\theta_H) \;=\; \sum_{t=0}^{\infty} \gamma^t \delta^t (\varphi_H - \varphi_L)^t \Bigg[(\theta_H - \theta_L) q_t(L^t) \\ + \delta(1-\gamma)(\rho_H - \rho_L) \mathbb{E}_g\big[B(\beta_t^s) \,|\, L^t\big] \Big) \Bigg] . \quad (5.5)$$

It follows by the definition of $U_0(\theta_0)$ that the on path expected discounted payments on period 1 satisfies

$$\mathbb{E}_\theta \left[\sum_{t=0}^{\infty} \gamma^t \delta^t \theta_t q_t(\theta^t) \,\middle|\, \theta_0 \right] - U_0(\theta_0) \;=\; \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} \gamma^t \delta^t p_t(\theta^t) \,\middle|\, \theta_0 \right]$$

It has been shown that $\text{IR}(\theta_0)$ binds, hence for a low type substitute $U_0(\theta_L) = 0$, whereas for a high type the expression derived in (5.5). Finally, substitute the expected discounted transfers on the objective function of ($\mathcal{P}$) to obtain ($\mathcal{P}'$).

To complete the proof note that by definition the transfers offered to a high type make him indifferent between deviating and not, after every history θ^{t-1}. Hence for the derived solution to be implementable it suffices that $\text{IC}(\theta_L^t)$ is satisfied. This is

$$U_t(\theta_L^t) \;\geq\; \theta_L q_t(\theta_H^t) - p_t(\theta_H^t) + \gamma\, \delta\, \varphi_L \widehat{U}_{t+1}(\theta_H, \theta_H, \theta_H^t) \\ +\; \delta\,(1-\gamma)\, \rho_L \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big] ,$$

where $\widehat{U}_{t+1}(\theta_L, \theta_L, \theta_H^t) \;=\; U_{t+1}(\{\theta_H^t, \theta_L\}) \;=\; 0$ has already been used on the continuation value on the right hand side. Substitute $U_t(\theta_L^t) = 0$ on the left hand side, $\widehat{U}_{t+1}(\theta_H, \theta_H, \theta_H^t) = U_{t+1}(\{\theta_H^t, \theta_H\})$ on the right one, rearrange and add the rest of the parts of $U_t(\theta_H^t)$ to obtain

$$(\theta_H - \theta_L) q_t(\theta_H^t) + \gamma\, \delta\, (\varphi_H - \varphi_L) U_{t+1}(\{\theta_H^t, \theta_H\}) \\ +\; \delta\,(1-\gamma)\,(\rho_H - \rho_L) \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big] \;\geq\; U_t(\theta_H^t)$$

Finally, substitute the recursive expression of $U_t(\theta_H^t)$ as it appears on (5.4), and note that by definition $U_{t+1}(\{\theta_H^t, \theta_H\}) = U_{t+1}^H(\theta_H^t)$ and $U_{t+1}(\{\theta_L^t, \theta_H\}) = U_{t+1}^H(\theta_L^t)$ to obtain ($\mathcal{P}_c$).

□

Proof of Corollary 1.1. The third line of ($\mathcal{P}'$) represents the information rents, which only affect the production of the L^t histories. Hence, point-wise maximisation on any other history simple optimises its first line, which represents the surplus from production, and gives the first-best level of effort. In contrast, for every t the production relevant payoff that corresponds to the L^t history is

$$\Pr(L^t)\gamma^t\delta^t \Bigg\{ \theta_L q_t(L^t) - \frac{q_t(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{\mu_0(\varphi_H - \varphi_L)^t}{\Pr(L^t)} (\theta_H - \theta_L) q_t(L^t) \Bigg\} \\ =\; \Pr(L^t)\gamma^t\delta^t \Bigg\{ \xi_t\, q_t(L^t) - \frac{q_t(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \Bigg\} ,$$

the point-wise maximisation of which gives $(\xi_t)^\epsilon$. Substitute the derived point-wise optimal quantities in $(\mathcal{P}_c)$ to obtain

$$(\theta_H-\theta_L)\Big[(\theta_H)^\epsilon-(\xi_t)^\epsilon+\sum_{t'=t+1}^{\infty}\big[(\varphi_H-\varphi_L)\gamma\delta\big]^{t'-t}\Big((\theta_L)^\epsilon-(\xi_t)^\epsilon\Big)\Big]\ \geq\ \delta(\rho_H-\rho_L)$$
$$\times\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)\Big\{\mathbb{E}_g\big[B(\beta^s_{t'})\,|\,\theta^t_L L^{t'}_{t+1}\big]-\mathbb{E}_g\big[B(\beta^s_{t'})\,|\,\theta^t_H L^{t'}_{t+1}\big]\Big\},\tag{5.6}$$

for all L^t histories, and for the remaining ones substitute ξ_t with θ_L. Note that $(\theta_L)^\epsilon\geq(\xi_t)^\epsilon$. Hence the left hand side above is bigger than $(\theta_H-\theta_L)[(\theta_H)^\epsilon-(\theta_L)^\epsilon]$. In addition, the right hand side of (5.6) is smaller than

$$\delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)\mathbb{E}_g\big[B(\beta^s_{t'})\,|\,\theta^t_L L^{t'}_{t+1}\big]$$
$$\leq\ \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)B(0)$$
$$\leq\ \delta(\rho_H-\rho_L)B(0)\ =\ \delta(\rho_H-\rho_L)b^{1+\epsilon}(\theta_H-\theta_L)\,(\theta_L)^\epsilon,$$

where the first inequality follows from noting that $B(\cdot)$ is decreasing, and the second because getting $B(0)$ in period $t{+}1$ for sure is better than any other realisation of τ. Hence combining the two equations together gives (1.4). □

Proof of Lemma 1.2. For brevity denote the event $(\tau=t)$ as (t) in the probabilities below. In $(\mathcal{G}_t)$ change the order of the summations and multiply the probabilities to obtain

$$\sum_s B(\beta^s_t)\Big\{\sum_{\theta^t}\big[\Pr(s,t\,|\,\theta^t)\Pr(\theta^t)\Pr(\theta_t=\theta_H|\,\theta_t)\big]$$
$$-\mu_0(\varphi_H-\varphi_L)^t(\rho_H-\rho_L)\Pr(s,t\,|\,L^t)\Big\}.\tag{5.7}$$

Moreover, note that

$$\sum_{\theta^t}\Pr(s,t\,|\,\theta^t)\Pr(\theta^t)\Pr(\theta_t=\theta_H|\theta_t)$$
$$=\ \phi_H\sum_{\theta^t_H}\Pr(s,t\,|\,\theta^t_H)\Pr(\theta^t_H)+\phi_L\sum_{\theta^t_L}\Pr(s,t\,|\,\theta^t_L)\Pr(\theta^t_L)$$

To transform the above equation note that for the period t high types

$$\Pr(\theta_t=\theta_H\,|\,s,t)\ =\ \frac{\sum_{\theta^t_H}\Pr(s,t\,|\,\theta^t_H)\Pr(\theta^t_H)}{\Pr(s,t)}\ \Leftrightarrow\ \mu^s_t\Pr(s,t)\ =\ \sum_{\theta^t_H}\Pr(s,t\,|\,\theta^t_H)\Pr(\theta^t_H)$$

Similarly, for the period t low types

$$(1-\mu_t^s)\,\Pr(s,t) \;=\; \sum_{\theta_L^t} \Pr(s,t\,|\,\theta_L^t)\,\Pr(\theta_L^t)$$

Finally, for the history L^t

$$\Pr(s,t\,|\,L^t) \;=\; \frac{\Pr(s,t,L^t)}{\Pr(L^t)} \;=\; \lambda_t^s\,\frac{\Pr(s,t)}{\Pr(L^t)} \;=\; \frac{\lambda_t^s}{(1-\varphi_L)^t}\,\frac{\Pr(s,t)}{1-\mu_0}$$

Hence, substitute the above in (5.7) to obtain

$$\begin{aligned}
&\sum_s B(\beta_t^s)\Big\{\rho_H\mu_t^s\Pr(s,t)+\rho_L(1-\mu_t^s)\Pr(s,t)-\psi_t\lambda_t^s\Pr(s,t)\Big\}\\
&\quad=\sum_s B(\beta_t^s)\Big\{\rho_H\mu_t^s+\rho_L(1-\mu_t^s)-\psi_t\lambda_t^s\Big\}\Pr(s,t)\;=\;\sum_s B(\beta_t^s)\big(\beta_t^s-\psi_t\lambda_t^s\big)g_t(s)\Pr(t)\,.
\end{aligned} \tag{5.8}$$

□

To shorten the statement of the subsequent lemma, let D denote the domain of $J_t(\mu,\lambda)$, and define its following subsets

$$\begin{aligned}
\overline{D} &\equiv \big\{(\mu,\lambda)\in D : \mu=0,\text{ or }\lambda=0,\text{ or }\mu+\lambda=1\big\}\,,\\
D_0 &\equiv \big\{(\mu,\lambda)\in D : \mu\geq\mu^*\big\} \quad\text{and}\quad D_I\equiv \overline{D}^c\cap D_o\,.
\end{aligned}$$

Thus $\overline{D}$ denotes the boundary of the domain, D_0 the subset of posteriors for which $J_t(\mu,\lambda)$ is flat, i.e. equal to zero, and D_I that of interior points on which $J_t(\mu,\lambda)$ is not flat.

Lemma 5.2. *$J_t(\mu,\lambda)$ is flat for all $\mu\in D_0$. Also, it is neither concave, nor convex for all $(\mu,\lambda)\in D_I$, and for an immediate termination $D_I=\{\emptyset\}$.*

- *On the boundary $(\mu=0)$: it is linear and decreasing on λ.*
- *On the boundary $(\lambda=0)$, and when $\Psi_t<\frac{\theta_L}{\theta_H}$ also on $(\mu+\lambda=1)$:*
  - *It changes monotonicity at most once, and if $\mu^*>0$ it falls to μ^* from above.*
  - *If $\epsilon\leq 1$, then it is strictly concave on $[0,\mu^*]$.*
  - *If $\epsilon>1$, then it is strictly concave on $[0,\mu_i^{**}]$, and strictly convex on $[\mu_i^{**},\mu^*]$, where*

$$\mu_i^{**}=\max\left\{0\,,\,\frac{\beta_i^{**}-\rho_L}{\rho_H-\rho_L}\right\}\quad\text{and}\quad \beta_i^{**}\;\equiv\;\frac{\theta_L}{\theta_H}\,\frac{2\big(1-\Psi_i\big)+(\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)\frac{\Psi_i}{\theta_L/\theta_H}}{2(1-\Psi_i)+(\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)}\,.$$

- *On the boundary $(\mu+\lambda=1)$, when $\Psi_t\geq\frac{\theta_L}{\theta_H}$: It is negative and strictly increasing for all $\mu<\mu^*$ Otherwise, it is equal to zero. In addition,*

– *If* $\epsilon \geq 1$*, then it is strictly concave on* $[0, \mu^*]$*.*

– *If* $\epsilon < 1$ *and* $(1-\epsilon)(1-\frac{\theta_L}{\theta_H}) \geq 2(1-\Psi_t)$*, then its strictly convex on* $[0, \mu^*]$*.*

– *If* $\epsilon < 1$ *and* $(1-\epsilon)(1-\frac{\theta_L}{\theta_H}) < 2(1-\Psi_t)$ *it is strictly concave on* $[0, \mu_i^{**}]$*, and strictly convex on* $[\mu_i^{**}, \mu^*]$*.*

Proof of Lemma 5.2. The time subscripts are suppressed. It is copied here from (5.1) that for every $\beta \leq \theta_L/\theta_H$:

$$B'(\beta) = -\frac{B(\beta)}{1-\beta}\,\epsilon\,\frac{\theta_H-\theta_L}{\theta_L-\beta\theta_H} \quad \text{and} \quad B''(\beta) = \frac{B'(\beta)}{1-\beta}\left(2+(1-\epsilon)\frac{\theta_H-\theta_L}{\theta_L-\beta\theta_H}\right),$$

Hence, for $\mu \leq \mu^*$ differentiating and re-arranging gives that

$$\begin{aligned}\frac{\partial J(\mu,\lambda)/\partial\mu}{\rho_H-\rho_L} = B'(\beta)(\beta-\psi\lambda)+B(\beta) \Rightarrow \frac{\partial^2 J(\mu,\lambda)/\partial\mu^2}{(\rho_H-\rho_L)^2} &= B''(\beta)(\beta-\psi\lambda)+2B'(\beta)\\ &= \frac{B'(\beta)}{1-\beta}\left[2(1-\psi\lambda)+(1-\epsilon)(b-\psi\lambda)\frac{\theta_H-\theta_L}{\theta_L-\beta\,\theta_H}\right].\end{aligned}$$

Likewise,

$$\frac{\partial J(\mu,\lambda)}{\partial\lambda} = -\psi B(\beta), \qquad \frac{\partial^2 J(\mu,\lambda)}{\partial\lambda\,\partial\mu} = -\psi(\rho_H-\rho_L)B'(\beta), \quad \text{and} \quad \frac{\partial^2 J(\mu,\lambda)}{\partial\lambda^2} = 0\,.$$

To prove that $J(\mu,\lambda)$ is neither concave nor convex on any of its interior points, that is not on the flat side of its domain, it suffices to show that its Hessian matrix is indefinite. This is given by

$$D^2J(\mu,\lambda) = \begin{pmatrix} \frac{\partial^2 J}{\partial\mu^2} & \frac{\partial^2 J}{\partial\mu\partial\lambda} \\ \cdot & \frac{\partial^2 J}{\partial\lambda^2}\end{pmatrix}$$

Hence, its determinant is $|D^2J| = -\left(\frac{\partial^2 J}{\partial\mu\partial\lambda}\right)^2 < 0$, from which it follows that it is indefinite.

To prove the statements for the boundaries $(\mu+\lambda=1)$ and $(\lambda=0)$, define the following linear combination of (μ',λ') and (μ'',λ''), for $w \in [0,1]$ and $\mu'' \neq \mu'$,

$$\begin{pmatrix}\bar{\mu}\\ \bar{\lambda}\end{pmatrix} = (1-w)\begin{pmatrix}\mu'\\ \lambda'\end{pmatrix} + w\begin{pmatrix}\mu''\\ \lambda''\end{pmatrix}.$$

This implies that

$$\begin{aligned} w = \frac{\bar{\mu}-\mu'}{\mu''-\mu'} \Rightarrow \bar{\lambda} &= \frac{\lambda''-\lambda'}{\mu''-\mu'}(\bar{\mu}-\mu')+\lambda' = \frac{\lambda''-\lambda'}{\mu''-\mu'}\left(\frac{\bar{b}-\rho_L}{\rho_H-\rho_L}-\mu'\right)+\lambda'\\ \Rightarrow (\bar{b}-\psi\,\bar{\lambda}) &= \bar{b}\underbrace{\left(1-\frac{\psi}{\rho_H-\rho_L}\frac{\lambda''-\lambda'}{\mu''-\mu'}\right)}_{\equiv\zeta} - \psi\underbrace{\left[\lambda'-\frac{\lambda''-\lambda'}{\mu''-\mu'}\left(\frac{\rho_L}{\rho_H-\rho_L}+\mu'\right)\right]}_{\equiv\zeta'}\end{aligned}$$

Let $\Psi \equiv \zeta'/\zeta$ and note that for the subsets $(\lambda = 0)$ and $(\mu + \lambda = 1)$

$$\begin{aligned}
\begin{pmatrix} \mu' & \mu'' \\ \lambda' & \lambda'' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} \zeta \\ \zeta' \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \begin{pmatrix} 1 + \frac{\psi}{\rho_H - \rho_L} \\ 1 + \frac{\rho_L}{\rho_H - \rho_L} \end{pmatrix} \\
\Psi &= \quad 0 \quad , \quad \frac{\psi \rho_H}{\rho_H - \rho_L + \psi},
\end{aligned}$$

respectively. In addition, substituting $\psi = \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t (\rho_H - \rho_L)$ I get that

$$\frac{\psi \rho_H}{\rho_H - \rho_L + \psi} = \frac{\mu_0 \rho_H}{(1-\mu_0)\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^{-t} + \mu_0} \leq \mu_0 \rho_H < 1.$$

Now the characterisation of $J(\mu, \lambda)$ on its boundaries can be obtained. Note that by moving $w \in [0, 1]$ I essentially move $J(\mu, \lambda)$ on the two specified sides. Moreover, because μ is a linear transformation of w its monotonicity and concavity changes at the same values of μ, and as a result of β. Hence, define

$$\begin{aligned}
\bar{J}(w) \equiv J\big[\bar{\mu}(w), \bar{\lambda}(w)\big] = \zeta\, B(\bar{\beta})(\bar{\beta} - \Psi)\,, \quad &\textit{where} \quad \bar{\beta} = \bar{\mu}(\rho_H - \rho_L) + \rho_L \\
&\text{and} \quad \bar{\mu} = w(\mu'' - \mu') + \mu'\,.
\end{aligned}$$

Then algebra similar to that used to simplify the partial derivatives of $J(\mu, \lambda)$ implies that

$$\bar{J}''(w) = (\rho_H - \rho_L)^2 (\mu'' - \mu')^2 \zeta \frac{B'(\bar{\beta})}{1 - \bar{\beta}} \left[2(1-\Psi) + (1-\epsilon)(\bar{\beta} - \Psi)\frac{\theta_H - \theta_L}{\theta_L - \bar{\beta}\,\theta_H}\right].$$

Hence, to find the set of $\bar{\beta}$'s for which $\bar{J}(w)$ is convex solve

$$\begin{aligned}
\bar{J}''(w) \geq 0 \;\Leftrightarrow\; 2(1-\Psi) + (1-\epsilon)(\bar{\beta} - \Psi)\frac{\theta_H - \theta_L}{\theta_L - \bar{\beta}\,\theta_H} \leq 0 \\
\Leftrightarrow\; \beta\left[2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\right] \geq 2(1-\Psi)\frac{\theta_L}{\theta_H} - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\Psi.
\end{aligned} \tag{5.9}$$

Next, four cases are considered. First suppose that $\epsilon \leq 1$ and $\theta_L/\theta_H \geq \Psi$, which implies

$$\frac{\theta_L}{\theta_H} \geq \Psi \;\Rightarrow\; \begin{cases} (1-\Psi) \geq \left(1 - \frac{\theta_L}{\theta_H}\right) & \Rightarrow \quad (1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right) \geq 0 \\ (1-\Psi)\frac{\theta_L}{\theta_H} \geq \left(1 - \frac{\theta_L}{\theta_H}\right)\Psi & \Rightarrow \quad (1-\Psi)\frac{\theta_L}{\theta_H} - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\Psi \geq 0 \end{cases}$$

Hence, (5.9) becomes

$$\beta \geq \frac{\theta_L}{\theta_H} \frac{2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)} \geq \frac{\theta_L}{\theta_H},$$

which is never satisfied on $[0, \theta_L/\theta_H)$. Second, suppose that $\epsilon > 1$ and $\theta_L/\theta_H \geq \Psi$, then (5.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H} \frac{2(1-\Psi) + (\epsilon-1)\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi) + (\epsilon-1)\left(1 - \frac{\theta_L}{\theta_H}\right)}$$

where the right hand side is equal to θ_L/θ_H for $\theta_L/\theta_H = \Psi$ and strictly less than it and positive for $\theta_L/\theta_H > \Psi$. Hence, in the former subcase it is always concave in $[0, \theta_L/\theta_H]$, while in the latter there is a point in this interval above which it becomes strictly convex. Third, suppose that $\epsilon \geq 1$ and $\theta_L/\theta_H < \Psi$. Then the second line of (5.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H} \frac{2(1-\Psi) + (\epsilon-1)\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi) + (\epsilon-1)\left(1 - \frac{\theta_L}{\theta_H}\right)} \;\geq\; \frac{\theta_L}{\theta_H},$$

Hence, similar to the first case $\bar{J}(w)$ is always concave on its non-flat part. Forth, suppose that $\epsilon < 1$ and $\theta_L/\theta_H < \Psi$. Consider the subcase where $2(1-\Psi) \leq (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)$, which implies that $2(1-\Psi)\frac{\theta_L}{\theta_H} \leq (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\Psi$. If the first inequality holds with equality, then (5.9) is trivially satisfied. Otherwise, it becomes

$$\beta \;\leq\; \frac{\theta_L}{\theta_H} \frac{(1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H} - 2(1-\Psi)}{(1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right) - 2(1-\Psi)},$$

the right hand side of which is strictly bigger than θ_L/θ_H. Hence, in this subcase the function is always convex in $[0, \theta_L/\theta_H]$. Finally, consider the subcase where $2(1-\Psi) > (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)$, for which (5.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H} \frac{2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)},$$

the right hand side of which is strictly smaller than θ_L/θ_H. Hence, in this subcase there exists a point in $[0, \theta_L/\theta_H)$ above which $\bar{J}(w)$ is strictly convex and below strictly concave. □

Proof of Proposition 1.2. The statement for the interior of the domain of $J_t(\mu, \lambda)$, for the first, and for third bullet point follow immediately from Lemma 5.2. To obtain the second let

$$\bar{J}_i(\mu) \;\equiv\; \widetilde{J}_i\Big(\mu(\rho_H - \rho_L) + \rho_L\Big), \quad \text{where} \quad \widetilde{J}_i(\beta) \;=\; \zeta_i B(\beta)(\beta - \Psi_i)$$

and instead of (1.6) solve

$$\widetilde{J}_i(\hat{\mu}_i) + \widetilde{J}_i'(\hat{\beta}_i)(\rho_H - \hat{\beta}_i) \;=\; 0\,, \tag{5.10}$$

in $[\Psi_i, \theta_L/\theta_H]$. It is ease to so that

$$\hat{\mu}_i \;\equiv\; \max\left\{\frac{\hat{\beta}_i - \rho_L}{\rho_H - \rho_L}\,, 0\right\}.$$

To solve (5.10) note that

$$\widetilde{J}_i(\beta) \;=\; \zeta_i B(\beta)(\beta-\Psi_i) \;\Rightarrow\; \widetilde{J}_i(\beta) \;=\; \zeta_i B(\beta)+\zeta_i B'(\beta)(\beta-\Psi_i)$$
$$= \zeta_i \frac{B(\beta)}{1-\beta}\left[1-\beta-\epsilon(\theta_H-\theta_L)\frac{\beta-\Psi_i}{\theta_L-\beta\theta_H}\right]$$

Hence, (5.10) equivalently becomes

$$(\hat{\beta}_i-\Psi_i)(1-\hat{\beta}_i)+\left[1-\hat{\beta}_i-\epsilon(\theta_H-\theta_L)\frac{\hat{\beta}_i-\Psi_i}{\theta_L-\hat{\beta}_i\theta_H}\right](\rho_H-\hat{\beta}_i) \;=\; 0$$
$$\Leftrightarrow \quad (1-\hat{\beta}_i)(\rho_H-\Psi_i) \;=\; \epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\hat{\beta}_i-\Psi_i}{\frac{\theta_L}{\theta_H}-\hat{\beta}_i}(\rho_H-\hat{\beta}_i) \tag{5.11}$$
$$\Leftrightarrow \quad \frac{\theta_L}{\theta_H}-\hat{\beta}_i\left(1+\frac{\theta_L}{\theta_H}\right)+\hat{\beta}_i^2 \;=\; \frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\left[\hat{\beta}_i(\rho_H+\Psi_i)-\hat{\beta}_i^2-\Psi_i\rho_H\right]$$

which after cancelling out terms and rearranging becomes

$$\omega_2\hat{\beta}_i^2-\omega_1\hat{\beta}_i+\omega_0 \;=\; 0\,, \quad \text{where} \quad \omega_0 \equiv \frac{\theta_L}{\theta_H}+\Psi_i\rho_H\frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\,,$$
$$\omega_1 \;\equiv\; 1+\frac{\theta_L}{\theta_H}+\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\rho_H+\Psi_i}{\rho_H-\Psi_i}\,, \quad \text{and} \quad \omega_2 \equiv 1+\frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\,. \tag{5.12}$$

In addition, it has already been shown that in the relevant parametric case $\bar{J}(\beta)$ changes its monotonicity and concavity at most once for $\beta<\theta_L/\theta_H$. Hence, because the function is initially concave, it is the smaller root that is the relevant one, while the bigger exists because of the shape that $B(\beta)$ would have for $\beta>\theta_L/\theta_H$ if it wasn't becoming flat there. Hence, the solution is given by

$$\hat{\beta}_i \;\equiv\; \frac{\omega_1-\sqrt{(\omega_1)^2-4\omega_0\omega_2}}{2\omega_2} \tag{5.13}$$

To prove the statement on its monotonicity with respect to Ψ_i equivalently rewrite the second line of (5.11) as

$$\hat{\beta}_i-1+\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\hat{\beta}_i-\Psi_i}{\rho_H-\Psi_i}\frac{\rho_H-\hat{\beta}_i}{\frac{\theta_L}{\theta_H}-\hat{\beta}_i} \;=\; 0\,, \tag{5.14}$$

where it follows from the above discussion that for the relevant solution given in (5.13) it has to be that $\Psi_i<\hat{\beta}_i<\frac{\theta_L}{\theta_H}\le\rho_H$. Hence the partial derivative of the left hand side of (5.14) with respect to $\hat{\beta}_i$ is

$$\frac{\partial LHS(5.14)}{\partial\hat{\beta}_i} \;=\; 1+\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\left[\frac{1}{\rho_H-\Psi_i}\frac{\rho_H-\hat{\beta}_i}{\frac{\theta_L}{\theta_H}-\hat{\beta}_i}+\frac{\hat{\beta}_i-\Psi}{\rho_H-\Psi_i}\frac{\rho_H-\frac{\theta_L}{\theta_H}}{(\frac{\theta_L}{\theta_H}-\hat{\beta}_i)^2}\right] \;>\; 0\,,$$

whereas that with respect to Ψ_i is

$$\frac{\partial LHS(5.14)}{\partial \Psi_i} \;=\; \epsilon \left(1-\frac{\theta_L}{\theta_H}\right) \frac{\rho_H-\hat{\beta}_i}{\frac{\theta_L}{\theta_H}-\hat{\beta}_i} \frac{\hat{\beta}_i-\rho_H}{(\rho_H-\Psi_i)^2} \;<\; 0\,.$$

Then the implicit function theorem gives that $\partial\hat{\beta}_i/\partial\Psi_i > 0$. □

Proof of Corollary 1.2 . The restriction $\varphi_H = 1-\varphi_L = 1$ implies that $\mu_t^s+\lambda_t^s = 1$, as a history where a high type changes to a low one never occurs. Hence it follows from Lemma 5.2 that

$$J_t(\mu,\lambda) \;=\; \bar{J}_t(\mu) \;=\; \left(1+\frac{\psi_t}{\rho_H-\rho_L}\right) B(\beta)\,(\beta-\Psi_t)\ ,$$

where to make notation more compact the shorthand $\beta = (\rho_H-\rho_L)\mu+\rho_L$ is used. Substitute $\varphi_H = 1-\varphi_L = 1$ to get that $\Psi_t = \frac{\mu_0}{1-\mu_0}$ and $\psi_t = \mu_0$, and $\rho_H = 1-\rho_L = 1$, which gives that $\beta = \mu$, to obtain

$$\bar{J}_t(\mu) \;=\; \bar{J}_0(\mu) \;=\; \frac{1}{1-\mu_0}B(\mu)(\mu-\mu_0)$$

Note that $\mu_t = \mu_0$. Then an argumentation identical to that used in the proof of Corollary 4 shows that no information provision is optimal. □

Proof of Proposition 1.3. First, it is shown that for all $(\mu,\lambda)\in D$ there exist points $(\mu',0)\in(\lambda=0)$ and $(\mu'',1-\mu'')\in(\mu+\lambda=1)$ such that

$$\mathcal{J}_t(\mu,\lambda) \;=\; \frac{\mu-\mu''}{\mu'-\mu''}\,\bar{\mathcal{J}}_f(\mu') \;+\; \frac{\mu'-\mu}{\mu'-\mu''}\,\bar{\mathcal{J}}_t(\mu'')\,. \tag{5.15}$$

To prove this first take point $(\mu_i,\lambda_i)\in D$ and let weight $\bar{\omega}_i$ be the one that gives this as a linear combination of $(\mu_i,0)$ and $(\mu_i,1-\mu_i)$. Then this solves

$$\bar{\omega}_i(1-\mu_i)+(1-\bar{\omega}_i)0 \;=\; \lambda_i$$

where $\bar{\omega}_i\in[0,1]$ since $\mu_i+\lambda_i\le 1$. Therefore,

$$\begin{aligned}&\bar{\omega}_i J_t(\mu_i,1-\mu_i)+(1-\bar{\omega}_i)J_t(\mu_i,0)\\ &=\; B(\beta_i)\beta_i-[\bar{\omega}_i(1-\mu_i)+(1-\bar{\omega}_i)0]\psi_t \;=\; J_t(\mu_i,\lambda_i)\,,\end{aligned}$$

which implies that $J_t(\mu_i,\lambda_i)$ can always be obtained as a linear combination of the value of J_t on two corresponding points on the boundaries $(\lambda=0)$ and $(\mu+\lambda=1)$.

Next, note that the concave closure $\mathcal{J}_t(\mu,\lambda)$ on every $(\mu,\lambda)\in D$ is a linear combination of J_t over a subset of D, call it $D(\mu,\lambda)$. Take any point $(\mu_i,\lambda_i)\in D(\mu,\lambda)$ that is also interior, and substitute $J_t(\mu_i,\lambda_i)$ with an additional linear combination between $(\mu_i,0)$ and $(\mu_i,1-\mu_i)$, while keeping the weight that multiplies $J_t(\mu_i,\lambda_i)$ constant. This leaves the value of $\mathcal{J}_t(\mu,\lambda)$ unchanged, as its follows from the above discussion that

$$\begin{aligned}J_t(\mu,\lambda) \;&=\; \sum_{(\mu_j,\lambda_j)\in D(\mu,\lambda)} \omega_j\, J_t(\mu_j,\lambda_j) \;=\; ...+\omega_i J_t(\mu_i,\lambda_i)\\ &=\; ...+\omega_i\bar{\omega}_i J_t(\mu_i,1-\mu_i)+\omega_i(1-\bar{\omega}_i)J_t(\mu_i,0)\,.\end{aligned}$$

Repeating the same process for all interior points of $D(\mu,\lambda)$ gives that $\mathcal{J}_t(\mu,\lambda)$ can be written as a linear combination of J_t over points belonging to $(\lambda=0)$ and $(\mu+\lambda=1)$ exclusively.

To prove (5.15) suppose that it does not hold for a point $(\mu,\lambda)\in D$. Then by maintaining the same probabilities of the posteriors to be on each of the two boundaries, $\Pr(\lambda_t^s=0)$ and $\Pr(\mu_t^s+\lambda_t^s=1)$, and changing the conditional probabilities so that the two corresponding conditional expectations are equal to the concave closures $\bar{\mathcal{J}}_f(\mu)$ and $\bar{\mathcal{J}}_t(\mu)$, respectively, the whole expectation increases, which leads to a contradiction.

Therefore, $\mathcal{J}_t(\mu,\lambda)$ can always be expressed as a linear combination of its value on the two boundaries. To find the correct weight for each set of points (μ,λ), $(\mu',0)$, and $(\mu'',1-\mu'')$ solve

$$\begin{pmatrix}\mu\\ \lambda\end{pmatrix} = \omega\begin{pmatrix}\mu'\\ 0\end{pmatrix} + (1-\omega)\begin{pmatrix}\mu''\\ 1-\mu''\end{pmatrix} \;\Rightarrow\; \omega = \frac{\mu-\mu''}{\mu'-\mu''}\,.$$

Hence to fully characterise $\mathcal{J}_t(\mu,\lambda)$ it remains to pin down μ' and μ''. To do this let x be the slope of the line that connects $(\mu',0)$, (μ,λ), and $(\mu'',1-\mu'')$. Then for $\mu\neq\mu'$, which also implies $\mu''\neq\mu'$, this is equal to:

$$x = \frac{\lambda-0}{\mu-\mu'} = \frac{1-\mu''-0}{\mu''-\mu'} \;\Rightarrow \begin{cases}\lambda & = x(\mu-\mu')\\ 1-\mu'' & = x(\mu''-\mu')\end{cases} \;\Rightarrow \begin{cases}\mu' & = \mu-\frac{\lambda}{x}\\ \frac{1-\mu'}{1-\mu''} & = 1+\frac{1}{x}\end{cases}\,.$$

This in turn implies

$$\frac{\mu'-\mu}{\mu'-\mu''} = \frac{\lambda}{1-\mu''} \quad \text{and}$$
$$\frac{\mu-\mu''}{\mu'-\mu''} = 1-\frac{\mu'-\mu}{\mu'-\mu''} = 1-\frac{\lambda}{1-\mu'}\frac{1-\mu'}{1-\mu''} = 1-\frac{\lambda}{1-\mu'}\left(1+\frac{1}{x}\right),$$

which gives that

$$\begin{aligned}(1-\mu')\frac{\mu-\mu''}{\mu'-\mu''} &= 1-\mu'-\lambda\left(1+\frac{1}{x}\right)\\ &= 1-\mu+\frac{\lambda}{x}-\lambda\left(1+\frac{1}{x}\right) = 1-\mu-\lambda\,.\end{aligned}$$

Then define

$$\begin{aligned}\widehat{\mathcal{J}}_t(x;\mu,\lambda) &\equiv \frac{\mu-\mu''}{\mu'-\mu''}\bar{\mathcal{J}}_f(\mu') + \frac{\mu'-\mu}{\mu'-\mu''}\bar{\mathcal{J}}_t(\mu'')\\ &= (1-\mu')\frac{\mu-\mu''}{\mu'-\mu''}\frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + (1-\mu'')\frac{\mu'-\mu}{\mu'-\mu''}\frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''}\\ &= (1-\mu-\lambda)\frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + \lambda\frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''}\,,\end{aligned}$$

and note that it follows from the above argumentation that

$$\mathcal{J}_t(\mu,\lambda) = \max_x\ \widehat{\mathcal{J}}_t(x;\mu,\lambda) \quad \text{s.t} \quad \mu',\mu''\in[0,1] \tag{5.16}$$

To write the constrain in terms of x solve for μ'', which is given by

$$x = \frac{1-\mu''-\lambda}{\mu''-\mu} \;\Leftrightarrow\; \mu'' = \frac{1-\lambda+x\mu}{1+x}\,.$$

As a result, for μ' and μ'' to be in the interval $[0,1]$ it has to be that

$$0 \leq \mu - \frac{\lambda}{x} \leq 1 \;\Rightarrow\; x \in \left(-\infty\,,\, -\frac{\lambda}{1-\mu}\right] \cup \left[\frac{\lambda}{\mu}\,,\, +\infty\right)$$
$$0 \leq \frac{1-\lambda+x\mu}{1+x} \leq 1 \;\Rightarrow\; x \in \left(-\infty\,,\, -\frac{1-\lambda}{\mu}\right] \cup \left[-\frac{\lambda}{1-\mu}\,,\, +\infty\right),$$

respectively. Then using that $1 > \mu + \lambda$ gives that the intersection of the above two sets is

$$x \in \left(-\infty\,,\, -\frac{1-\lambda}{\mu}\right] \cup \left[\frac{\lambda}{\mu}\,,\, +\infty\right),$$

which is the desired form for the constrain of (5.16). □

6 Proofs for continuous types

To avoid repetition, the subsequent lemma provides a sufficient and necessary condition for implementation that will be applied to the contracts offered by both S_1 and P_b.

Lemma 6.1 (Implementation). *For given price $p : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ suppose that the payoff of a θ type buyer, when reporting $\hat{\theta}$, is*

$$\widehat{V}(\hat{\theta}, \theta) \equiv v(\hat{\theta}, \theta) - p(\hat{\theta}) \tag{6.1}$$

where $v : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is absolute continuous in the second variable with weak derivative $v_2 : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$. Then truthful reporting is implementable only if v_2 is increasing in the first variable. In addition, if this holds then the price

$$p(\theta) = v(\theta, \theta) - \int_{\underline{\theta}}^{\theta} v_2(x, x)\,\mathrm{d}x \tag{6.2}$$

ensures that truthful reporting in optimal.

Proof of Lemma 6.1*.* First, necessity is proven. Suppose that truthful reporting is optimal and let $V(\theta) = \widehat{V}(\theta, \theta)$, then for any $\theta_1, \theta_2 \in [\underline{\theta}, \overline{\theta}]$ such that $\theta_1 < \theta_2$:

$$V(\theta_2) \geq \widehat{V}(\theta_1, \theta_2) = V(\theta_1) + \int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_1, \theta)\,\mathrm{d}\theta$$
$$V(\theta_1) \geq \widehat{V}(\theta_2, \theta_1) = V(\theta_2) - \int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_2, \theta)\mathrm{d}\theta$$

where the subscript 2 indicates the partial derivative with respect to the second entry. Rearranging the two inequalities and combining them gives

$$\int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_2, \theta)\,\mathrm{d}\theta \geq \int_{\theta_1}^{\theta_2} \widehat{V}_2^s(\theta_1, \theta)\,\mathrm{d}\theta.$$

As this has to hold for any choice of θ_1 and θ_2, as defined above, it follows that $v_2(\hat{\theta},\theta) = \widehat{V}_2(\hat{\theta},\theta)$ has to be non-decreasing on $\hat{\theta}$. Second, sufficiency is proven. Suppose $\hat{\theta} < \theta$, then

$$\begin{aligned} v(\hat{\theta},\theta) - p(\hat{\theta}) &= v(\hat{\theta},\theta) - v(\hat{\theta},\hat{\theta}) + \int_{\underline{\theta}}^{\hat{\theta}} v_2(x,x)\,\mathrm{d}x \\ &= \int_{\hat{\theta}}^{\theta} v_2(\hat{\theta},x)\,\mathrm{d}x + \int_{\underline{\theta}}^{\hat{\theta}} v_2(x,x)\,\mathrm{d}x \\ = \int_{\hat{\theta}}^{\theta} \Big\{ v_2(\hat{\theta},x) - v_2(x,x) \Big\}\,\mathrm{d}x &+ \int_{\underline{\theta}}^{\theta} v_2(x,x)\,\mathrm{d}x \;\leq\; \int_{\underline{\theta}}^{\theta} v_2(x,x)\,\mathrm{d}x \;=\; v(\theta,\theta) - p(\theta) \end{aligned}$$

As a result reporting $\hat{\theta} = \theta$ is no worse than any $\hat{\theta} < \theta$. The proof for $\hat{\theta} > \theta$ is similar hence it is omitted. □

Proof of Lemma 2.1. S_2's revenue for given choice of policies $p_2(\theta_2)$ and $q_2(\theta_2)$ is

$$\int_{\underline{\theta}_2}^{\bar{\theta}_2} \Big\{ p_2(\theta_2) - c[q_2(\theta_2)] \Big\}\,\mathrm{d}F_2^s(\theta_2), \tag{6.3}$$

The buyer's payoff, when reporting $\hat{\theta}_2$ instead of his actual type θ_2, is

$$\widehat{V}_2^s(\hat{\theta}_2,\theta_2) \;=\; b\,\theta_2 q_2(\hat{\theta}_2) - p_2(\hat{\theta}_2)\,.$$

It is ease to argue that Theorem 2 of Milgrom and Segal (2002) applies in this setting. Hence their envelop theorem gives that

$$\frac{\mathrm{d}V_2^s(\theta_2)}{\mathrm{d}\theta_2} \;=\; b\,q_2(\theta_2)\,.$$

As a result, integrating and equating with $V_2^s(\theta_2) = \widehat{V}_2^s(\theta_2,\theta_2)$ gives

$$b\int_{\underline{\theta}_2}^{\theta_2} q_2(x)\,\mathrm{d}x + V_2^s(\underline{\theta}_2) \;=\; b\,\theta_2 q_2(\theta_2) - p_2(\hat{\theta}_2)\,.$$

Then the objective function of (2.2) follows from substituting in (6.3) the price $p_2(\theta_2)$, as given from the above equality, applying Fubini's Theorem, and setting $V_2^s(\underline{\theta}_2) = 0$.

Finally, note that it follows from Lemma 6.1 that a sufficient and necessary condition for the choice of $q_2(\theta_2)$ to be implementable is that it is non-decreasing. □

Proof of Proposition 2.1. Under static types $\bar{V}_2^s(\theta_1) = V_2^s(\theta_1)$. Hence S_1's ex ante payoff from the buyer's contract with S_2 can equivalently be rewritten as follows

$$\begin{aligned} \int_{\underline{\theta}_1}^{\bar{\theta}_1} \mathbb{E}_g\big[\bar{V}_2^s(\theta_1)\,|\,\theta_1\big]\mathrm{d}F_1(\theta_1) &= \int_{\underline{\theta}_1}^{\bar{\theta}_1} \sum_s \big\{\bar{V}_2^s(\theta_1) g(s\,|\,\theta_1)\big\}\,\mathrm{d}F_1(\theta_1) \\ = \sum_s \int_{\underline{\theta}_1}^{\bar{\theta}_1} \bar{V}_2^s(\theta_1) f_1^s(\theta_1)\mathrm{d}\theta_1\, g(s) &= \sum_s \int_{\underline{\theta}_1}^{\bar{\theta}_1} [1 - F_1^s(\theta_1)]\,\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\mathrm{d}\theta_1\, g(s) \\ = \int_{\underline{\theta}_1}^{\bar{\theta}_1} \sum_s \mu_1^s(\theta_1)\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} g(s\,|\,\theta_1)\,\mathrm{d}F_1(\theta_1) &= b\int_{\underline{\theta}_0}^{\bar{\theta}_0} \sum_s \big\{\mu_1^s(\theta_1) q_1^s(\theta_1) g(s\,|\,\theta_1)\,\big\}\mathrm{d}F_1(\theta_1) \end{aligned}$$

where the last equality is due to the fact that $\theta_1 = \theta_2$ under static types. It follows from (6.3) that under no information provision the quantity implemented by S_2 is

$$b^{\epsilon} \max\left\{0\,,\,\theta_1 - \mu_1(\theta_1)\right\}^{\epsilon},$$

but since $\mu_1(\theta_1)$ is non-increasing by assumption, then whenever $\mu_1(\underline{\theta}_1) > \underline{\theta}_1$ there exists a set $[\underline{\theta}_1, \theta_1^{+}]$ of positive measure for which the supplied quantity is zero. Consider a binary signal $s \in \{s^-, s^+\}$ such that

$$\begin{aligned} g(s^- \,|\, \theta_1) &= 1\,, \quad \text{if} \quad \theta_1 \leq \theta^+ \\ g(s^+ \,|\, \theta_1) &= 1\,, \quad \text{if} \quad \theta_1 > \theta^+ \end{aligned}$$

which essentially reveals the corresponding set in which the buyer's type belongs. It is ease to show that the corresponding inverse hazard rates are

$$\mu_1^-(\theta_1) \;=\; \begin{cases} \frac{F_1(\theta^+) - F_1(\theta_1)}{f_1(\theta_1)} & ,\ \text{if} \ \ \theta_1 \leq \theta^+ \\ \text{not defined} & ,\ \text{if} \ \ \theta_1 > \theta^+ \end{cases} \qquad \text{and} \quad \mu_1^+(\theta_1) \;=\; \begin{cases} \text{not defined} & ,\ \text{if} \ \ \theta_1 \leq \theta^+ \\ \mu_1(\theta_1) & ,\ \text{if} \ \ \theta_1 > \theta^+ \end{cases}$$

Note that both of them are non-increasing in θ_1, hence after the realisation of each of them S_2 implements the corresponding point-wise optimal quantity. As a result, S_1's ex ante revenue under this binary signal becomes

$$b^{1+\epsilon} \int_{\underline{\theta}_1}^{\theta^+} \mu_1^-(\theta_1) \max\left\{0\,,\,\theta_1 - \mu_1^-(\theta_1)\right\}^{\epsilon} \mathrm{d}F_1(\theta_1) \\ +\; b^{1+\epsilon} \int_{\theta^+}^{\bar{\theta}_1} \mu_1(\theta_1) \max\left\{0\,,\,\theta_1 - \mu_1(\theta_1)\right\}^{\epsilon} \mathrm{d}F_1(\theta_1)$$

The second line is identical to S_1's ex ante payoff under no information provision, but the first would be zero instead. On the other hand, under the constructed binary signal there will be at least a few types in $[\underline{\theta}_1, \theta^+]$ that will be supplied a positive quantity, as $\mu_1^-(\theta^+) = 0$. Hence, for a subset of types close to θ^+ the supplied quantity will be strictly positive. □

Proof of Lemma 2.2. It is ease to argue that Theorem 2 of Milgrom and Segal (2002) applies in this setting. Hence their envelop theorem gives that

$$\frac{\mathrm{d}V_1(\theta_1)}{\mathrm{d}\theta_1} \;=\; q_1(\theta_1) + \mathbb{E}_g\left[\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\theta_1\right].$$

As a result, integrating and equating with $V_1(\theta_1) = \widehat{V}_1(\theta_1, \theta_1)$ gives

$$\int_{\underline{\theta}_1}^{\theta_1} \frac{\mathrm{d}V_1(x)}{\mathrm{d}x}\,\mathrm{d}x \;=\; \theta_1 q_1(\theta_1) - p_1(\theta_1) + \mathbb{E}_g\left[\bar{V}_2^s(\theta_1)\,|\,\theta_1\right].$$

Hence substitute in the objective function of (2.3) the price $p_1(\theta_1)$, as given from the above equality, apply Fubini's Theorem, and set $V_1(\underline{\theta}_1) = 0$ to obtain

$$\int_{\underline{\theta}_1}^{\bar{\theta}_1} \Bigg\{ q_1(\theta_1)\left[\theta_1 - \mu_1(\theta_1)\right] - c[q_1(\theta_1)] \\ + \mathbb{E}_g\left[\bar{V}_2^s(\theta_1) - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\theta_1\right] \Bigg\} \mathrm{d}F_1(\theta_1) \qquad (6.4)$$

Then the functional form of the point-wise optimal production follows from the first order condition of the above. Lemma 6.1 implies that for the pair (q_1^*, g) to be implementable it has to be that

$$\frac{\partial\,\widehat{V}_1(\hat\theta_1,\theta_1)}{\partial\,\theta_1} \;=\; q_1^*(\hat\theta_1) + \mathbb{E}_g\left[\,\frac{\mathrm{d}\bar V_1(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\hat\theta_1\right]$$

is non-decreasing in $\hat\theta_1$, from which condition (2.5) is derived. Note that $q_1^*(\hat\theta_1)$ is increasing in $\hat\theta_1$, while under no information provision the above expectation is constant with respect to it. Hence no information provision is implementable.

Finally, to derive the objective function of (2.4) rewrite the second line of (6.4) as follows

$$\begin{aligned}
&\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_g\left[\bar V_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar V_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\theta_1\right]\Big\}\mathrm{d}F_1(\theta_1)\\
&\qquad = \int_{\underline{\theta}_1}^{\overline{\theta}_1}\sum_s\left\{\bar V_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar V_2^s(\theta_1)}{\mathrm{d}\theta_1}\right\}g(s|\theta_1)f_1(\theta_1)\,\mathrm{d}\theta_1\\
&\qquad = \sum_s\int_{\underline{\theta}_1}^{\overline{\theta}_1}\left\{\bar V_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar V_2^s(\theta_1)}{\mathrm{d}\theta_1}\right\}f_1^s(\theta_1)\,\mathrm{d}\theta_1\,g(s)\\
&\qquad\qquad = \sum_s\int_{\underline{\theta}_1}^{\overline{\theta}_1}\left\{\frac{1-F_1^s(\theta_1)}{f_1^s(\theta_1)} - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\right\}\frac{\mathrm{d}\bar V_2^s(\theta_1)}{\mathrm{d}\theta_1}f_1^s(\theta_1)\mathrm{d}\theta_1 g(s)\,.
\end{aligned}$$

□

Proof of Proposition 2.2. First, it is shown that $\frac{\mathrm{d}\bar V_2^s(\theta_1)}{\mathrm{d}\theta_1} \geq 0$. This function has been defined as

$$\bar V_2^s(\theta_1) \;=\; \mathbb{E}_{\theta_2}\big[V_2^s(\theta_2)\,|\,\theta_1, s\big]\,,$$

where it follows from (2.1) that $V_2^s(\theta_2)$ is non-decreasing, because $\frac{\mathrm{d}V_2^s(\theta_2)}{\mathrm{d}\theta_2} = b\,q_2^s(\theta_2) \geq 0$. But then since higher values of θ_1 induce a conditional CDF $F_2(\cdot\,|\,\theta_1)$ that FOSD lower values, it follows that $\bar V_2^s(\theta_1)$ is non-decreasing in θ_1.

Next, it is shown that under any deterministic signal it has to be that $\mu_1^s(\theta_1) \leq \mu_1(\theta_1)$. To do this fix a partition $\{\Theta_1^s\}_{s\in S}$ of $[\underline{\theta}_1, \overline{\theta}_1]$ and note that the probability of each signal s to be realised is

$$g(s) \;=\; \int_{\theta_1\in\Theta_1^s} f_1(\theta_1)\,\mathrm{d}\theta_1\,.$$

As a result, the posterior density and CDF are

$$f_1^s(\theta_1) \;=\; \begin{cases} f_1(\theta_1)/g(s) & ,\text{ if }\ \theta_1\in\Theta_1^s\\ 0 & ,\text{ if }\ \theta_1\notin\Theta_1^s\end{cases} \qquad\text{and}\qquad F_1^s(\theta_1) \;=\; \int_{x\leq\theta_1\,\cap\,x\in\Theta_1^s} f_1(x)\,\mathrm{d}x\,\frac{1}{g(s)}\,.$$

respectively. Hence whenever $\mu_1^s(\theta_1)$ is defined, that is for $\theta_1\in\Theta_1^s$, this is given by

$$\begin{aligned}
\mu_1^s(\theta_1) \;&=\; \frac{1-F_1^s(\theta_1)}{f_1^s(\theta_1)} \;=\; \int_{x>\theta_1\,\cap\,x\in\Theta_1^s} f_1(x)\,\mathrm{d}x\,\frac{1}{f_1(\theta_1)}\\
&\leq\; \int_{x>\theta_1} f_1(x)\,\mathrm{d}x\,\frac{1}{f_1(\theta_1)} \;=\; \mu_1(\theta_1)
\end{aligned}$$

To finish the proof note that the objective function of (2.4) under the deterministic signal that uses the partition $\{\Theta_1^s\}_{s\in S}$ becomes

$$\sum_s \int_{\theta_1 \in \Theta_1^s} \Big\{ \mu_1^s(\theta_1) - \mu_1(\theta_1) \Big\} \frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} \, \mathrm{d}F_1(\theta_1) \;\leq\; 0$$

But under no information provision the same objective function is equal to zero, and this is implementable, hence this signalling structure is optimal among deterministic signals. ☐

Proof of Proposition 2.3. The proof is an extension of the treatment of continuous types undertaken by Calzolari and Pavan (2006), which can be found in the Appendix of their paper. A discussion similar to theirs can demonstrated that the incentives to provide information to S_2 are the strongest when S_1 captures all the expected benefit from this information provision. Moreover, in this case S_1's payoff is equivalent, up to a constant transformation, to as if she was integrated with S_2. Hence, if it can be shown that no information provision is optimal in this case, then the same is true when S_1 captures none of this expected benefit from information provision.

To do this set $\varphi = 1$ and suppose momentarily that S_1 and S_2 were integrated and solve the corresponding dynamic mechanism design problem. Denote the realised history $\{\theta_1, \theta_2\}$ by θ^2, and let $\theta^1 = \theta_1$. Then S_1 solves:

$$\mathbb{E}_{\theta^t} \left[\max_{p_t, q_t} \sum_{t=1}^{2} \left\{ p_t(\theta^t) - c\big[q_t(\theta^t)\big] \right\} \right] , \tag{6.5}$$

subject to the individual rationality constrains of period 1, and the incentive compatibility constrains of period 1 and period 2. First, the above problem is solved under the following restriction on the set of deviations used by the buyer.

- In period 1: he freely chooses the report $\hat{\theta}_1$
- In period 2: he is restricted to truthfully report if his type was redrawn, or not.

Let $\widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1)$ denote the payoff of an buyer that reported $\hat{\theta}_1$ in period 1, his type was redrawn, and he subsequently reported $\hat{\theta}_2$, while his actual type was θ_2.

$$\widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1) \;=\; b\, \theta_2\, q_2(\hat{\theta}_1, \hat{\theta}_2) - p_2(\hat{\theta}_1, \hat{\theta}_2)$$

In addition, let $U_2(\theta_2; \hat{\theta}_1) \equiv \widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1)$ be the corresponding value under truthful reporting of θ_2. Note that the actual value of θ_1 is irrelevant in terms of his incentives to report $\hat{\theta}_2$. Hence, incentive compatibility implies that

$$U_2(\theta_2; \hat{\theta}_1) \;=\; \max_{\hat{\theta}_2 \in [0,1]} \; \widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1)\,, \quad \text{for all} \quad \hat{\theta}_1 \in [0,1]$$

In all of the subsequent discussion Theorem 2 of Milgrom and Segal (2002) applies. Hereafter, it will simple be invoked as *envelop theorem*. Then the envelop theorem gives that

$$\frac{\mathrm{d}U_2(\theta_2; \hat{\theta}_1)}{\mathrm{d}\theta_2} \;=\; b\, q_2(\hat{\theta}_1, \theta_2)$$

Set $U_2(0\,;\hat{\theta}_1) = 0$ and note that

$$\begin{aligned} U_2(\theta_2;\hat{\theta}_1) \;&=\; \int_0^{\theta_2} \frac{\mathrm{d}U_2(x;\hat{\theta}_1)}{\mathrm{d}x}\,\mathrm{d}x \;\Leftrightarrow \\ b\,\theta_2\,q_2(\hat{\theta}_1,\theta_2) - p_2(\hat{\theta}_1,\theta_2) \;&=\; \int_0^{\theta_2} b\,q_2(\hat{\theta}_1,\theta_2)\,\mathrm{d}x \end{aligned}$$

Since this hold for any $\hat{\theta}_1$, it also holds for $\hat{\theta}_1 = \theta_1$. Hence the part of (6.5) that follows a redraw of θ_2, after a realisation of θ_1, can be rewritten as

$$\begin{aligned} &\mathbb{E}_{\theta_2}\Big[p_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big]\Big] \\ &\qquad = \int_0^1 \left\{ b\,\theta_2\,q_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big] - \int_0^{\theta_2} b\,q_2(\theta_1,\theta_2)\,\mathrm{d}x \right\} f_1(\theta_2)\,\mathrm{d}\theta_2 \\ &\qquad\qquad = \int_0^1 \left\{ b\,\big[\theta_2 - \mu_1(\theta_2)\big]\,q_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big] \right\} f_1(\theta_2)\,\mathrm{d}\theta_2 \qquad (6.6) \end{aligned}$$

Next, denote the expected payoff of the buyer in period 1 under such a deviation by

$$\begin{aligned} \widehat{U}_1(\hat{\theta}_1,\theta_1) \;=\; \theta_1 q_1(\hat{\theta}_1) - p_1(\hat{\theta}_1) + \rho\Big(\,b\,\theta_1 q_2(\hat{\theta}_1,\hat{\theta}_1) - p_2(\hat{\theta}_1,\hat{\theta}_1)\,\Big) \\ + (1-\rho)\mathbb{E}_{\theta_2}\Big[\,\widehat{U}_2(\hat{\theta}_2,\theta_2;\hat{\theta}_1)\,\Big]\,, \end{aligned}$$

and let $U_1(\theta_1) = \widehat{U}_1(\theta_1,\theta_1)$. Then under truthful reporting the envelop theorem implies that

$$\frac{\mathrm{d}U_1(\theta_1)}{\mathrm{d}\theta_1} \;=\; q_1(\theta_1) + \rho b\,q_2(\theta_1,\theta_1)$$

As a result, using the same manipulations as above I obtain that the part of the (6.5) that corresponds to the realisation of θ_1 and of it not being redrawn can be written as

$$\big[\theta_1 - \mu_1(\theta_1)\big]q_1(\theta_1) - c\big[q_1(\theta_1)\big] + \rho\Big\{b\,\big[\theta_1 - \mu_1(\theta_1)\big]q_2(\theta_1,\theta_1) - c\big[q_2(\theta_1,\theta_1)\big]\Big\} \qquad (6.7)$$

Then point-wise maximisation of (6.6) and (6.7) gives that

$$q_t^*(\theta_t) \;=\; \max\Big\{0\,,\,b^t\big[\theta_t - \mu_1(\theta_t)\big]\Big\}^{\epsilon} \qquad (6.8)$$

This solves the optimisation of (6.5) under a restriction on the buyer's action space. But it is ease to show using Lemma 6.1 that this supply schedule is also implementable without the imposed restriction. This in turn gives that S_1 can achieve the same payoff in the problem where the buyer has access to his full action space, as in the restricted one. Hence, the supply schedule (6.8) is a solution to her optimisation problem (6.5).

Switching back to the non-integrated S_1 and S_2 case, note that (6.8) is what the latter would chose under no information provision. Hence, it follows from the discussion in the beginning of this proof that no information provision is optimal for her. □

7 Proofs for Moral Hazard

Proof of Lemma 3.1. The dependence on (τ, s) is dropped. The revelation principle applies and since the agent's type is static, it is without loss to focus on contracts that pay period wage $w^b(\theta, y_t)$ in $t \in \{\tau, .., \infty\}$. Moreover, observe that using the reported type, a perfect estimate of the effort can be deducted by P_b. Hence, any misalignment between this estimate and the recommended effort can be punished strongly enough for the agent to mask it. As a result a report $\hat{\theta}$ implies choice of effort

$$\hat{e}^b(\hat{\theta}, \theta) \;=\; e^b(\hat{\theta}) \cdot \frac{\hat{\theta}}{\theta}.$$

For simplicity drop the dependence on y_t, as this will occur only off path and write the reported type as a subscript for $w^b(\theta)$ and $e^b(\theta)$. P_b's payoff maximisation problem is essentially a static one.

$$\begin{aligned}
\max_{e,w} \quad & \beta\left(b\,\theta_H e^b_H - w^b_H\right) + (1-\beta)\left(b\,\theta_L e^b_L - w^b_L\right) \\
\text{s.t. } (\mathrm{IR}_L) \quad & w^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; 0 \\
(\mathrm{IR}_H) \quad & w^b_H - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; 0 \\
(\mathrm{IC}_L) \quad & w^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w^b_H - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} \\
(\mathrm{IC}_H) \quad & w^b_H - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}},
\end{aligned}$$

where both the constrains and the objective function are written in per period payoff. Assuming that (IR_L) does not bind leads to a contradiction. Subsequently, this can be used to show that (IC_H) has to also bind. Hence the above simplifies to the unconstrained maximisation problem

$$\max_{e} \;\beta\left(b\,\theta_H\,e^b_H - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right) + (1-\beta)\left(b\,\theta_L\,e^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\,\frac{1-\beta\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}{1-\beta}\right)$$

The objective function is concave, hence the first order conditions give that

$$e^b(\theta_H) \;=\; (b\,\theta_H)^{\epsilon} \quad \text{and} \quad e^b(\theta_L) \;=\; b^{\epsilon}\cdot\left(\frac{(1-\beta)\,\theta_L}{1-\beta\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}\right)^{\epsilon}.$$

This is implementable as (IC_L) equivalently becomes

$$\begin{gathered}
w^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w^b_H - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} - 1\right] \Leftrightarrow \\
w^b_L - \frac{(e^b_L)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w^b_L - \frac{(e^b_L)^2}{2}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} - \frac{(e^b_H)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} - 1\right] \Leftrightarrow \\
\theta_H e^b_H \;\geq\; \theta_L e^b_L
\end{gathered}$$

which is satisfied for the derived effort choices. Because the IR_L binds the low type's period payoff is zero. The high type's period payoff can be obtained using the (IR_L) and (IC_H) constrains, which give that

$$
\begin{aligned}
w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;=\; w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} \;=\; \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[1-\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}\right] \\
= \frac{(b\,\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left[1-\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}\right]\left(\frac{1-\beta}{1-\beta\cdot\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}\right)^{1+\epsilon}.
\end{aligned}
$$

Hence a constant stream of the above payoff up to infinity gives $B(\beta)$. Moreover,

$$
B'(\beta) \;=\; -(1+\epsilon)K\left(\frac{1-\beta}{1-\beta\kappa}\right)^{\epsilon}\frac{1-\kappa}{(1-\beta\kappa)^2} \;=\; -B(\beta)\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)} \;<\; 0
$$

for all $\beta \in [0,1)$. In addition,

$$
\begin{aligned}
B''(\beta) \;&=\; -(1+\epsilon)K(1-\kappa)\frac{-\epsilon(1-\beta)^{\epsilon-1}(1-\beta\kappa)^{\epsilon+2}+\kappa(\epsilon+2)(1-\beta)^{\epsilon}(1-\beta\kappa)^{\epsilon+1}}{(1-\beta\kappa)^{2(\epsilon+2)}} \\
&=\; B'(\beta)\left(\frac{\kappa(\epsilon+2)}{1-\beta\kappa}-\frac{\epsilon}{1-\beta}\right) > 0 \Leftrightarrow \frac{\kappa(\epsilon+2)}{1-\beta\kappa} < \frac{\epsilon}{1-\beta} \Leftrightarrow \beta > 1-\epsilon\frac{1-k}{k}.
\end{aligned}
$$

□

Proof of Proposition 3.1. Lemma 5.1 applies in this setting. Hence it is without loss to only consider one-shot deviations. In those a θ^t agent type reports truthfully θ^{t-1}, potentially misreports θ_t as $\hat\theta_t$, and subsequently switches back to truthful reporting. Hereafter, $\text{IC}(\theta^t)$ will refer exclusivity to the incentive compatibility constrains obtained under one-shot deviations. To maintain a compact notation, let $\hat\theta^t = \{\theta^{t-1}, \hat\theta_t\}$ denote a history of truthful reports up to $t-1$ followed by a potential misreport $\hat\theta_t$. In addition, denote a generic history θ^{t-1} followed by $\theta_t = \theta_H$ as θ_H^t, and similarly define $\hat\theta_L^t$. Then the payoff of a θ^t agent type under a one shot deviation is

$$
\begin{aligned}
\widehat{U}_t(\hat\theta_t, \theta_t, \theta^{t-1}) \;=\; w_t^a(\hat\theta^t) - \frac{e_t^a(\hat\theta^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\hat\theta_t}{\theta_t}\right)^{1+\frac{1}{\epsilon}} \;+\; f_t(\hat\theta^t)\gamma\delta\,\mathbb{E}_\theta\big[\widehat{U}_{t+1}(\theta_{t+1}, \theta_{t+1}, \hat\theta^t)\,|\,\theta_t\big] \\
+\; [1-f_t(\hat\theta^t)\gamma]\,\delta\;\Pr(\theta_{t+1}=\theta_H\,|\,\tau=t,\theta_t)\,\mathbb{E}_g\big[B(\beta_t^s)\,|\,\hat\theta^t\big]
\end{aligned}
$$

In period t the agent obtains the wage of the reported type $\hat\theta_t$, but since his type is θ_t he actually has to mask this deviation which is why there is this adjustment on the cost of effort. The probability of continuation $\gamma f_t(\hat\theta^t)$ is only a function of $\hat\theta^t$, however under one shot deviations the agent's type is truthfully reported in $\widehat{U}_{t+1}$ and the expectation over it depends on the actual type θ_t. Similarly, the probability of the agent to be a high type in P_b's contract, $\Pr(\theta_{t+1}=\theta_H\,|\,\tau=t,\theta_t)$, is only a function of the realised type θ_t. In contrast, the distribution of the signal s, provided by P_a to P_b, is contingent on the reported type $\hat\theta^t$.

Let the on path payoff of a θ^t type agent be given by $U_t(\theta^t) = \widehat{U}_t(\theta_t, \theta_t, \theta^{t-1})$, then the corresponding individual rationality and incentive compatibility constrains become

$$\begin{aligned} \text{IR}(\theta^t) \quad U_t(\theta^t) \;&\geq\; 0 \\ \text{IC}(\theta^t) \quad U_t(\theta^t) \;&\geq\; \widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1}) \end{aligned} \tag{7.1}$$

where the agent's outside option is zero because if he decides to terminate the contract in $t < \tau$, then P_a can ensure that P_b will not approach the agent. Doing so is always beneficial for P_a, because it lowers the agent's outside option. Next, consider the following problem

$$\begin{aligned} (\mathcal{P}^H) \qquad \max_{w,e,f,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1}) \gamma^t \delta^t \Big(\theta_t \, e_t^a(\theta^t) - w_t^a(\theta^t) \Big) \right] \\ \text{subject to IR}(\theta^t) \text{ and IC}(\theta^t), \end{aligned} \tag{7.2}$$

which is similar to ($\mathcal{P}$), but it ignores the individual rationality constrains of all high types $\text{IR}(\theta_H^t)$ and the incentive compatibility constrains of all low types $\text{IC}(\theta_L^t)$. Ignoring the former set of constrains is without loss of generality as $U_t(\theta_H^t) \geq U_t(\theta_L^t)$, however this is not true for the $\text{IC}(\theta_L^t)$ constrains. Despite that, if the solution of ($\mathcal{P}^H$) happens to also satisfy this set of constrains, then it is a solution of ($\mathcal{P}$).

The rest of the proof shows that in ($\mathcal{P}^H$), for every (w, e, f, g) there exists a w' such that the objective function under (w', e, f, g) is no less than under (w, e, f, g), and that both the $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ constrains bind. Solving for w' from the constrains and substituting in the objective function will give ($\mathcal{P}'$). As a result, I will have shown that if (e, f, g) is a solution of ($\mathcal{P}'$), then there exists w' such that (w', e, f, g) is also a solution of ($\mathcal{P}^H$). Finally, w' will be substituted in the previously ignored $\text{IC}(\theta_L^t)$ so that a sufficient condition is obtained for (w', e, f, g) be a solution of ($\mathcal{P}$), which is only in terms of policies (e, f, g).

The argument is recursive. Suppose that $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ bind for all periods up to and including t', but not for $t'+1$. For simplicity, denote $w_t^a(\theta_H^{t'})$ by w_H, $w_t^a(\theta_L^{t'})$ by w_L, and similarly use $\{w_{LL}, w_{HL}, w_{LH}, w_{HH}\}$ for the possible combinations up to $t'+1$. Moreover, adopt the same notational change for the IR and IC constrains. Suppose that IR_{HL} does not bind, then let

$$\big(\tilde{w}_H, \tilde{w}_{HH}, \tilde{w}_{HL}\big) \;=\; \big(w_H + \delta\varepsilon,\, w_{HH} - \varepsilon,\, w_{HL} - \varepsilon\big)$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is indifferent between the two contracts. The same argument works if IR_{LL} does not bind. Suppose instead that IC_{LH} does not bind, then let

$$\big(\tilde{w}_L,\, \tilde{w}_H,\, \tilde{w}_{LH}\big) \;=\; \big(w_L + \delta\varphi_L\varepsilon,\, w_H - \delta(\varphi_H - \varphi_L)\varepsilon,\, w_{LH} - \varepsilon\big),$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is actually better off. Finally, suppose that IC_{HH} does not bind, then let

$$\big(\tilde{w}_H,\, \tilde{w}_{HH}\big) \;=\; \big(w_H + \delta\varphi_H\varepsilon,\, w_{HH} - \varepsilon\big),$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is indifferent between the two contracts. Hence, if both $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ bind for all periods up to and including t', and $\text{IC}(\theta_L^t)$ is ignored, then there exists an alternative contract that implements the same policies, is not worse for P_a, and has all constrains binding up to $t'+1$. In addition, the regular one period argumentation shows that $\text{IR}(\theta_L^0)$ and $\text{IC}(\theta_H^0)$ have to bind, from which the recursive argument follows.

Hence, it is without loss to assume that $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ bind, which gives

$$-w_t^a(\theta_L^t) \;=\; -\frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} + f_t(\theta_L^t)\gamma\delta\varphi_L\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1-f_t(\theta_L^t)\gamma]\delta\rho_L\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]$$

and

$$\begin{aligned} U_t(\theta_H^t) \;=\; & w_t^a(\theta_L^t) - \frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}} \\ & + f_t(\theta_L^t)\gamma\delta\varphi_H\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1-f_t(\theta_L^t)\gamma]\delta\rho_H\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t], \end{aligned}$$

respectively. Note that in both equations above it has been used that $\widehat{U}_{t+1}(\theta_L,\theta_L,\theta_L^t) = U_{t+1}(\{\theta_L,\theta_L^t\}) = 0$, where the first equality follows from the ignored $\text{IC}(\{\theta_L,\theta_L^t\})$ and the second from the fact that $\text{IR}(\{\theta_L,\theta_L^t\})$ binds. Substitute the first line in the second to obtain that

$$\begin{aligned} U_t(\theta_H^t) \;=\; & \frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \\ & + f_t(\theta_L^t)\gamma\delta(\varphi_H-\varphi_L)\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1-f_t(\theta_L^t)\gamma]\delta(\rho_H-\rho_L)\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]. \end{aligned}$$

But because the $IC(\{\theta_L^t,\theta_H\})$ holds, I get that $U_{t+1}(\{\theta_L^t,\theta_H\}) = \widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t)$ and the same argument can be used repeatedly for any $t' > t+1$. Hence, substitute forward to obtain the functional form given for $U_t^H(\theta^{t-1})$. In particular, for period 0 that becomes

$$\begin{aligned} U_0(\theta_H) \;=\; & \sum_{t=0}^{\infty} f_0^{t-1}(L^{t-1})\gamma^t\delta^t(\varphi_H-\varphi_L)^t \\ & \times\left[\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) + \delta[1-f_t(L^t)](\rho_H-\rho_L)\mathbb{E}_g\big[B(\beta_t^s)\mid L^t\big]\Big)\right]. \end{aligned} \tag{7.3}$$

It follows by the definition of $U_0(\theta_0)$ that the on path expected discounted payments In period 0 satisfies

$$U_0(\theta_0) + \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t\frac{e_t^a(\theta^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\,\middle|\,\theta_0\right] \;=\; \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t w_t^a(\theta^t)\,\middle|\,\theta_0\right]$$

It has being shown that $\text{IR}(\theta_0)$ binds, hence for a low type substitute above $U_0(\theta_L) = 0$, while for a high type substitute the expression obtained in (5.5). Finally, substituting the expected discounted payments on the objective function gives ($\mathcal{P}'$).

To complete the proof note that by definition the wages given to a high type makes him indifferent between deviating or not, after every history θ^{t-1}. For a low type the IC(θ_L^t) becomes

$$U_t(\theta_L^t) \;\geq\; w_t^a(\theta_H^t) - \frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} + f_t(\theta_H^t)\gamma\,\delta\,\varphi_L U_{t+1}(\{\theta_H^t,\theta_H\}) \\ + \delta\,[1-f_t(\theta_H^t)\gamma]\,\phi_L \mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big]\,,$$

which after substituting $U_t(\theta_L^t)=0$ equivalently becomes

$$U_t(\theta_H^t) \;\leq\; \frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right) + \delta\,\Pr(\tau>t+1\,|\,\tau>t,\theta_H^t)\,(\varphi_H-\varphi_L)\,U_{t+1}(\theta_H^t,\theta_H) \\ + \delta\,\Pr(\tau=t+1\,|\,\tau>t,\theta_H^t)(\rho_H-\rho_L)\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big].$$

Finally, substituting the provided expression for $U_t^H(\theta^{t-1})$, and re-arranging gives ($\mathcal{P}_c$).

□

Proof of Corollary 3.1. The third line of ($\mathcal{P}'$) represents the information rents, which only affect the production of the L^t histories. Hence, point-wise maximisation on any other history simple optimises its first line, which represents the surplus from production, and gives the first-best level of effort. In contrast, for every t the production relevant payoff that corresponds to the L^t history is

$$\Pr(L^t)f_0^{t-1}(L^{t-1})\gamma^t\delta^t\left\{\theta_L e_t^a(L^t) - \frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{\mu_0(\varphi_H-\varphi_L)^t}{\Pr(L^t)}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right)\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right\} = \\ \Pr(L^t)f_0^{t-1}(L^{t-1})\gamma^t\delta^t\left\{\theta_L e_t^a(L^t) - \xi_t\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right\}$$

the point-wise maximisation of which gives θ_L/ξ_t. Hence substitute the derived effort in the value functions of ($\mathcal{P}_c$) to obtain

$$\frac{(\theta_H)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right) - \frac{e_t^*(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) + \left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \\ \times\sum_{t'=t+1}^{\infty}\big[(\varphi_H-\varphi_L)\gamma\delta\big]^{t'-t}\left(f_t^{t'-1}(\theta_H^t L_{t+1}^{t'-1})\frac{(\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}} - f_t^{t'-1}(\theta_L^t L_{t+1}^{t'-1})\frac{e_{t'}^*(\theta_L^t L_{t+1}^{t'})^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right) \\ \geq\; \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Big\{\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\big] \\ - \Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_H^t L_{t+1}^{t'})\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_H^t L_{t+1}^{t'}\big]\Big\} \quad (7.4)$$

Note that $(\theta_L)^\epsilon \geq e_t^*(\theta_L^t)$, hence the second line of (7.4) is non-negative under a non-decreasing termination policy. Hence, its left hand side is bigger than

$$\frac{(\theta_H)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right)-\frac{e_t^*(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \geq \frac{\theta_H^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right)-\frac{\theta_L^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right).$$

In addition, the right hand side of (7.4) is smaller than

$$\begin{aligned}
&\delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\left[(\varphi_H-\varphi_L)\delta\right]^{t'-t}\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})\mathbb{E}_g\left[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\right]\\
&\quad\geq\ \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\left[(\varphi_H-\varphi_L)\delta\right]^{t'-t}\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})B(0)\\
&\quad\geq\ \delta(\rho_H-\rho_L)B(0)\ =\ \delta(\rho_H-\rho_L)\frac{1-\kappa}{1-\delta}\frac{(b\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}},
\end{aligned}$$

where the first inequality follows from noting that $B(\cdot)$ is decreasing, and the second because getting $B(0)$ in period t+1 for sure is better than any other distribution for τ. Hence combining the two equations together gives the first sufficient condition. For $\epsilon = 1$, this becomes

$$\frac{(\theta_H^2-\theta_L^2)^2}{\theta_H^2\theta_L^2}\ \geq\ \frac{\delta\, b^s}{1-\delta}(\rho_H-\rho_L)(\theta_H^2-\theta_L^2)\frac{\theta_L^2}{\theta_H^2},$$

which after substituting $\kappa = \theta_L^2/\theta_H^2$ gives the derived sufficient condition. □

Proof of Lemma 3.3. The time subscripts are suppressed. Differentiating and rearranging gives that

$$\begin{aligned}
B'(\beta)\ &=\ K(1+\epsilon)\left(\frac{1-\beta}{1-\beta\kappa}\right)^{\epsilon}\frac{-(1-\beta\kappa)+\kappa(1-\beta)}{(1-\beta\kappa)^2}\\
&=\ -K(1+\epsilon)(1-\kappa)\frac{(1-\beta)^\epsilon}{(1-\beta\kappa)^{\epsilon+2}} = -B(\beta)\,\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)} < 0.
\end{aligned}$$

Also,

$$\begin{aligned}
B''(\beta)\ &=\ K(1+\epsilon)(1-\kappa)\frac{(1-\beta)^{\epsilon-1}}{(1-\beta\kappa)^{\epsilon+3}}\Big(\epsilon(1-\beta\kappa)-(\epsilon+2)\kappa(1-\beta)\Big)\\
&= -B'(\beta)\,\frac{\epsilon(1-\beta\kappa)-(\epsilon+2)\kappa(1-\beta)}{(1-\beta)(1-\beta\kappa)}.
\end{aligned}$$

As a result,

$$\frac{\partial J_t(\eta,\lambda)/\partial\eta}{\rho_H-\rho_L}\ =\ B'(\beta)(\beta-\psi\,\lambda)+B(\beta)\ =\ B(\beta)\left[1-(\beta-\psi\,\lambda)\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)}\right],$$

and

$$\frac{\partial^2 J_t(\eta,\lambda)/\partial\eta^2}{(\rho_H-\rho_L)^2} = B''(\beta)(\beta-\psi\,\lambda)+2B'(\beta)$$
$$= -B'(\beta)\left[\frac{\epsilon(1-\beta\kappa)-(\epsilon+2)\kappa(1-\beta)}{(1-\beta)(1-\beta\kappa)}(\beta-\psi\,\lambda)-2\right].$$

Finally,

$$\frac{\partial J_t(\eta,\lambda)}{\partial\lambda} = -\psi B(\beta), \qquad \frac{\partial^2 J_t(\eta,\lambda)}{\partial\lambda\partial\eta} = -\psi(\rho_H-\rho_L)B'(\beta), \qquad \frac{\partial^2 J_t(\eta,\lambda)}{\partial\lambda^2} = 0.$$

To prove that $J_t(\eta,\lambda)$ is neither concave nor convex on any of its interior points it suffices to show that its Hessian matrix is indefinite. This is given by

$$D^2 J_t(\eta,\lambda) = \begin{pmatrix} \frac{\partial^2 J}{\partial\eta^2} & \frac{\partial^2 J}{\partial\eta\partial\lambda} \\ \cdot & \frac{\partial^2 J}{\partial\lambda^2} \end{pmatrix}$$

Hence, its determinant is $|D^2 J| = -\left(\frac{\partial^2 J}{\partial\eta\partial\lambda}\right)^2 < 0$, from which it follows that it is indefinite.

To prove the statements for the boundaries $(\eta+\lambda=1)$ and $(\lambda=0)$, define the following linear combination of (η',λ') and (η'',λ''), for $w\in[0,1]$ and $\eta''\neq\eta'$,

$$\begin{pmatrix}\bar\eta\\ \bar\lambda\end{pmatrix} = (1-w)\begin{pmatrix}\eta'\\ \lambda'\end{pmatrix} + w\begin{pmatrix}\eta''\\ \lambda''\end{pmatrix}.$$

This implies that

$$w=\frac{\bar\eta-\eta'}{\eta''-\eta'} \Rightarrow \bar\lambda = \frac{\lambda''-\lambda'}{\eta''-\eta'}(\bar\eta-\eta')+\lambda' = \frac{\lambda''-\lambda'}{\eta''-\eta'}\left(\frac{\bar b-\rho_L}{\rho_H-\rho_L}-\eta'\right)+\lambda'$$
$$\Rightarrow (\bar\beta-\psi\,\bar\lambda) = \bar\beta\underbrace{\left(1-\frac{\psi}{\rho_H-\rho_L}\frac{\lambda''-\lambda'}{\eta''-\eta'}\right)}_{=\zeta}-\psi\underbrace{\left[\lambda'-\frac{\lambda''-\lambda'}{\eta''-\eta'}\left(\frac{\rho_L}{\rho_H-\rho_L}+\eta'\right)\right]}_{=\zeta'}$$

Let $\Psi=\zeta'/\zeta$ and note that for the subsets $(\lambda=0)$ and $(\eta+\lambda=1)$

$$\begin{pmatrix}\eta' & \eta''\\ \lambda' & \lambda''\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\begin{pmatrix}\zeta_0\\ \zeta_1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}1+\frac{\psi}{\rho_H-\rho_L}\\ 1+\frac{\rho_L}{\rho_H-\rho_L}\end{pmatrix}$$
$$\Psi = 0, \quad \frac{\psi\rho_H}{\rho_H-\rho_L+\psi},$$

respectively. In addition, substituting $\psi=\frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H-\varphi_L}{1-\varphi_L}\right)^t(\rho_H-\rho_L)$ I get that

$$\frac{\psi\rho_H}{\rho_H-\rho_L+\psi} = \frac{\mu_0\rho_H}{(1-\mu_0)\left(\frac{\varphi_H-\varphi_L}{1-\varphi_L}\right)^{-t}+\mu_0} \leq \mu_0\rho_H < 1.$$

Now the characterisation of $J_t(\eta,\lambda)$ on its boundaries can be obtained. Note that by moving $w \in [0,1]$ I essentially move $J_t(\eta,\lambda)$ on the two specified sides. Moreover, because η is a linear transformation of w its monotonicity and concavity changes at the same values of η, and as a result of β.

$$\begin{aligned} \bar{J}(w) \;&=\; J\big[\bar{\eta}(w),\bar{\lambda}(w)\big] \;=\; \zeta_0 B(\bar{\beta})(\bar{\beta}-\Psi)\,, \quad \textit{where} \quad \bar{\beta} \;=\; \bar{\eta}(\rho_H-\rho_L)+\rho_L \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad \bar{\eta} \;=\; w(\eta''-\eta')+\eta'\,. \end{aligned}$$

Then algebra similar to that used to simplify the partial derivatives of $J_t(\eta,\lambda)$ implies that

$$\begin{aligned} \bar{J}'(w) \;&=\; (\rho_H-\rho_L)(\eta''-\eta')\zeta_0 B(\bar{\beta})\left(1-(\bar{\beta}-\Psi)\,\frac{1+\epsilon}{(1-\bar{\beta})(1-\bar{\beta}\kappa)}\right) \\ \bar{J}''(w) \;&=\; (\rho_H-\rho_L)^2(\eta''-\eta')^2\zeta_0[-B'(\bar{\beta})] \\ &\qquad \times\left(\frac{\epsilon(1-\bar{\beta}\kappa)-(\epsilon+2)\kappa(1-\bar{\beta}))}{(1-\bar{\beta})(1-\bar{\beta}\kappa)}\,(\bar{\beta}-\Psi)-2\right) \end{aligned}$$

First, note that $\bar{J}'(w)\geq 0$ if and only if

$$(1-\bar{\beta})(1-\bar{\beta}\kappa) \;\geq\; (\bar{\beta}-\Psi)(1+\epsilon)(1-\kappa) \;\Leftrightarrow\; \big[1+\Psi(1+\epsilon)(1+\kappa)\big]-\big[2+\epsilon(1-\kappa)\big]\bar{\beta}+\kappa\bar{\beta}^2 \geq 0,$$

solving for the roots of the left hand side gives

$$\frac{2+\epsilon(1-\kappa)\pm\sqrt{[2+\epsilon(1-\kappa)]^2-4\kappa-4\kappa(1-\kappa)(1+\epsilon)\Psi}}{2\kappa}.$$

The roots are not necessarily real numbers, but when they are the (+) one is always greater than one as $\frac{2+\epsilon(1-\kappa)}{2\kappa} > \frac{1}{\kappa} \geq 1$. Also, it is ease to show that the (−) one is positive. Simplifying we get that

$$\bar{J}'(w)\geq 0 \Leftrightarrow \begin{cases} \forall\,\bar{\beta}\in[0,1] & \text{, if } \Psi \geq \frac{1}{\kappa}+\frac{\epsilon^2(1-\kappa)}{4(1+\epsilon)\kappa} \\ \bar{\beta}\leq\beta^*(\Psi) & \text{, if } \Psi \leq \frac{1}{\kappa}+\frac{\epsilon^2(1-\kappa)}{4(1+\epsilon)\kappa} \end{cases}.$$

However for $\Psi\in[0,1)$, which is the case here, only the second line is relevant. Similarly, I get that $\bar{J}'(w)\geq 0$ if and only if

$$\begin{gathered} 2(1-\bar{\beta})(1-\bar{\beta}\kappa) \;\leq\; \big\{[\epsilon(1-\kappa)-2\kappa]+2\kappa\bar{\beta}\big\}\,(\bar{\beta}-\Psi) \Leftrightarrow \\ 2-2(1+\kappa)\bar{\beta}+2\kappa\bar{\beta}^2 \leq [\epsilon(1-\kappa)-2\kappa](\bar{\beta}-\Psi)+2\kappa\bar{\beta}^2-2\kappa\Psi\bar{\beta} \Leftrightarrow \\ 2(1-\Psi\kappa)+(1-\kappa)\Psi\epsilon \;\leq\; \bar{\beta}\Big[2(1-\Psi\kappa)+(1-\kappa)\epsilon\Big] \end{gathered}$$

where the expression in the brackets of the right hand side is positive for $\Psi\in[0,1)$, hence this equivalently becomes

$$\bar{\beta} \;\geq\; \frac{2(1-\Psi\kappa)+(1-\kappa)\Psi\epsilon}{2(1-\Psi\kappa)+(1-\kappa)\epsilon},$$

where again for $\Psi\in[0,1)$ the right hand side is positive and smaller than one. In addition, note that the because $\bar{J}(w)$ is initially increasing it has a maximum on $\beta^*(\Psi)$, but this implies that the function is concave there, hence $\beta^*(\Psi)\leq\beta^{**}(\Psi)<1$. □

8 Proofs for Endogenous Termination

The main variables are

$$f_t = \mu_t x_t^H + (1-\mu_t) x_t^l\,, \qquad \eta_t = \frac{\mu_t(1-\gamma x_t^H)}{1-\gamma f_t}\,, \qquad \mu_{t+1} = \frac{\mu_t x_t^H + (1-\mu_t) x_t^l \varphi_L}{f_t}\,.$$

The following derivatives will be repeatedly used in the subsequent analysis. Derivatives of f_t:

$$\frac{\partial f_t}{\partial \mu_t} = x_t^H - x_t^l\,, \qquad \frac{\partial f_t}{\partial x_t^l} = 1-\mu_t\,, \qquad \frac{\partial f_t}{\partial x_t^H} = \mu_t\,.$$

Derivatives of η_t:

$$\frac{\partial \eta_t}{\partial \mu_t} = \frac{\eta_t^2}{\mu_t^2}\frac{1-\gamma x_t^l}{1-\gamma x_t^H} = \frac{\eta_t(1-\eta_t)}{\mu_t(1-\mu_t)}\,, \qquad \frac{\partial \eta_t}{\partial x_t^l} = \frac{\gamma(1-\mu_t)}{1-\gamma f_t}\eta_t\,, \qquad \frac{\partial \eta_t}{\partial x_t^H} = -\frac{\gamma \mu_t}{1-\gamma f_t}(1-\eta_t)\,.$$

Derivatives of μ_{t+1}:

$$\frac{\partial \mu_{t+1}}{\partial \mu_t} = (1-\varphi_L)\frac{x_t^H\, x_t^l}{f_t^2}\,, \qquad \frac{\partial \mu_{t+1}}{\partial x_t^l} = -(1-\varphi_L)\frac{(1-\mu_t)\mu_t}{f_t^2}x_t^H\,, \qquad \frac{\partial \mu_{t+1}}{\partial x_t^H} = \frac{\mu_t}{f_t}(1-\mu_{t+1})\,.$$

As argued in the main text, P_a's problem has the following recursive representation

$$V_t(\mu_t) = \max_{x_t^H, x_t^l} \Big\{ \mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t) + h_t^1(1-x_t^H) + h_t^0 x_t^H + l_t^1(1-x_t^l) + l_t^0 x_t^l \Big\}\,, \tag{8.1}$$

where h_t^1 and h_t^0 are the Lagrange multipliers for $0 \le x_t^H \le 1$, and l_t^1, l_t^0 are the corresponding multipliers for $0 \le x_t^l \le 1$. Let $v_t(x_t^H, x_t^l, \mu_t)$ denote the first line of the above objective function, that is the expression in the parenthesis, but without the constrains. The problem is solved under the following generic assumption.

Assumption. $\mathcal{J}_0$ is twice continuously differentiable and concave. Both u_H and u_l are positive. Also,

$$u_H > (1-\delta)\mathcal{J}_0(1) \qquad \text{and} \qquad \frac{u_H}{1-\delta\gamma} + \frac{\delta(1-\gamma)}{1-\delta\gamma}\mathcal{J}_0(1) > \mathcal{J}_0(0) + \mathcal{J}_0'(0). \tag{8.2}$$

The proof proceeds as follows. First, it is shown that $x_t^H = 1$ is optimal under the supposition that V_{t+1} is twice continuously differentiable and concave. Second, it is shown that in this case V_t is also twice continuously differentiable and concave. Third, the contraction mapping theorem is used to show that $V_t = V_{t+1} = V$, and that V is indeed concave and twice continuously differentiable. Forth, some sufficient conditions are provided for each of the three possible solutions for x_t^t (interior, $x_t^l = 0$, and $x_t^l = 1$) to be relevant.

There is actually one value on which the functional form of V_t and V_{t+1} can easily be derived, which is when the agent's current reputation is one.

Lemma 8.1. *For $\mu_t = 1$ it is strictly optimal to continue, $x_t^H(1) = 1$. Moreover,*

$$V(1) \;=\; \frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1)\,. \tag{8.3}$$

Proof. For $\mu_t = 1$ the principal's dynamic problem simplifies to

$$V(1) \;=\; \max_{x_t^H} \;\; u_H + \delta\gamma x_t^H V(1) + \delta(1-\gamma x_t^H)\mathcal{J}_0(1).$$

The payoff from always continuing $x_t^H = 1$ is

$$\frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1).$$

On the other hand, that of stopping $x_t^H = 0$ is $\mathcal{J}_0(1)$. Hence, no-stopping is proffered to stopping when

$$\frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1) \;>\; \mathcal{J}_0(1) \quad\Leftrightarrow\quad u_H \;>\; (1-\delta)\mathcal{J}_0(1),$$

which has been assumed to hold. □

This allows the derivation of the following result.

Lemma 8.2. *Suppose that $V_{t+1}(\mu_{t+1})$ is twice continuously differentiable and concave, then always continuing a high type, $x_t^H(\mu_t) = 1$, is strictly optimal for every $\mu_t \in (0,1]$.*

Proof. Differentiating gives

$$\begin{aligned}\frac{\partial v_t}{\partial x_t^H} \;&=\; \delta\gamma\mu_t V_{t+1}(\mu_{t+1}) + \delta\gamma\mu_t(1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) - \delta\gamma\mu_t\mathcal{J}_0(\eta_t) - \delta\gamma\mu_t(1-\eta_t)\mathcal{J}'_0(\eta_t)\\ &=\; \delta\gamma\mu_t\Big[\, V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) - \mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t)\,\Big]\end{aligned}$$

But note that

$$\begin{aligned}\frac{\partial}{\partial\mu_{t+1}}\Big[V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1})\Big] \;&=\; (1-\mu_{t+1})V''_{t+1}(\mu_{t+1}) \;\leq\; 0\\ \Rightarrow\quad V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) \;&\geq\; V_{t+1}(1)\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial\eta_t}\Big[-\mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t)\Big] \;&=\; -(1-\eta_t)\mathcal{J}''_0(\eta_t) \;\geq\; 0\\ \Rightarrow\quad -\mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t) \;&\geq\; -\mathcal{J}_0(0) - \mathcal{J}'_0(0).\end{aligned}$$

As a result,

$$\frac{\partial v_t}{\partial x_t^H} \;\geq\; \delta\gamma\mu_t\Big(V_{t+1}(1) - \mathcal{J}_0(0) - \mathcal{J}'_0(0)\Big) \;>\; 0,$$

because $V_{t+1}(1) = V(1)$, as identified in the previous lemma, which implies that the derivative $\partial v_t/\partial x_t^H$ has to be strictly positive. This in turn implies that it is always strictly optimal to set $x_t^H = 1$. □

As a result (8.1) simplifies to

$$V_t(\mu_t) = \max_{x_t^l} \Big\{ \mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t) + l_t^1(1-x_t^l) + l_t^0 x_t^l \Big\} \tag{8.4}$$

where

$$f_t = \mu_t + (1-\mu_t)x_t^l, \quad \eta_t = \frac{\mu_t(1-\gamma)}{1-\gamma f_t}, \quad \mu_{t+1} = \frac{\mu_t + (1-\mu_t)x_t^l\varphi_L}{f_t}. \tag{8.5}$$

Differentiating with respect to x_t^l gives

$$\begin{aligned}
\frac{\partial v_t}{\partial x_t^l} &= \delta\gamma(1-\mu_t)\Big(V_{t+1}(\mu_{t+1}) - (1-\varphi_L)\frac{\mu_t}{f_t}V'_{t+1}(\mu_{t+1}) - \mathcal{J}_0(\eta_t) + \eta_t\mathcal{J}_0'(\eta_t)\Big) = l_t^1 - l_t^0, \\
\frac{\partial^2 v_t}{\partial (x_t^l)^2} &= \delta\gamma(1-\mu_t)\Big((1-\varphi_L)^2\frac{\mu_t^2(1-\mu_t)}{f_t^2}V''_{t+1}(\mu_{t+1}) + \frac{\gamma(1-\mu_t)}{1-\gamma f_t}\eta_t^2\mathcal{J}_0''(\eta_t)\Big) \leq 0.
\end{aligned} \tag{8.6}$$

Moreover, it follows from the Envelop Theorem that

$$\begin{aligned}
V_t'(\mu_t) &= u_H - u_l + \delta\gamma(1-x_t^l)V_{t+1}(\mu_{t+1}) + \delta\gamma(1-\varphi_L)\frac{x_t^l}{f_t}V'_{t+1}(\mu_{t+1}) \\
&\quad - \delta\gamma(1-x_t^l)\mathcal{J}_0(\eta_t) + \delta(1-\gamma x_t^l)\frac{\eta_t}{\mu_t}\mathcal{J}_0'(\eta_t).
\end{aligned} \tag{8.7}$$

Substitute the first order condition from (8.6) in (8.7) to obtain

$$\begin{aligned}
V_t'(\mu_t)\mu_t - V_t(\mu_t) &= -u_l - \delta\gamma x_t^l V_{t+1}(\mu_{t+1}) - \delta(1-x_t^l\gamma)\mathcal{J}_0(\eta_t) \\
&\quad + \delta\gamma(1-\varphi_L)\frac{\mu_t x_t^l}{f_t}V'_{t+1}(\mu_{t+1}) + \delta(1-\gamma x_t^l)\eta_t\mathcal{J}_0'(\eta_t) \quad \Rightarrow \\
V_t'(\mu_t)\mu_t - V_t(\mu_t) &= -u_l + x_t^l\frac{l_t^0 - l_t^1}{1-\mu_t} - \delta\mathcal{J}_0(\eta_t) + \delta\eta_t\mathcal{J}_0'(\eta_t).
\end{aligned} \tag{8.8}$$

Lemma 8.3. *Suppose that $V_{t+1}(\mu_{t+1})$ is twice continuously differentiable and concave, then the same is true for $V_t(\mu_t)$. Moreover,*

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \geq 0 \quad \text{and} \quad \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \geq 0.$$

Proof. The first statement of the lemma follows trivially. The rest of the proof focuses on proving concavity and the two derivatives. First, suppose that the non-negative constrain binds so that

$$l_t^0 > 0 \Rightarrow \begin{cases} x_t^l = 0 \\ l_t^1 = 0 \end{cases},$$

then total differentiation, with respect to μ_t, on the last line of (8.8) gives

$$V_t''(\mu_t)\mu_t = -(u_H - u_l) + \delta\mathcal{J}_0''(\eta_t)\eta_t\frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \leq 0. \tag{8.9}$$

This is because for $x_t^l = 0$,

$$\eta_t \;=\; \frac{\mu_t(1-\gamma)}{1-\gamma\mu_t} \;\Rightarrow\; \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \;=\; (1-\gamma)\frac{1+\gamma\mu_t}{(1-\gamma\mu_t)^2} \;>\; 0.$$

Second, suppose that $x_t^l \in (0,1)$. Total differentiation on (8.8) gives (8.9) again, but now $x_t^l(\mu_t)$ is not known so the derivative of $\eta_t(\mu_t)$ cannot be calculated. Instead use total differentiation, with respect to μ_t, on the foc of x_t^l on (8.6), which for an interior solution gives

$$(1-\varphi_L)\frac{\mu_t}{f_t}V''_{t+1}(\mu_{t+1})\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;=\; \mathcal{J}''_0(\eta_t)\eta_t\frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t}, \tag{8.10}$$

hence for this case it suffices to show that $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$. This will imply that the left hand side of (8.10) is negative, which in turn will give the same for the right hand side, from which it will also follow that $\mathrm{d}\eta_t/\mathrm{d}\mu_t \geq 0$. Simple differentiation on the function form of μ_{t+1} gives

$$\mu_{t+1} \;=\; \frac{\mu_t + (1-\mu_t)x_t^l\varphi_L}{\mu_t+(1-\mu_t)x_t^l} \quad\Rightarrow\quad \frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;=\; \frac{1-\varphi_L}{f_t^2}\left(x_t^l - \mu_t(1-\mu_t)\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t}\right).$$

For an interior solution, the derivative $\mathrm{d}x_t^l/\mathrm{d}\mu_t$ can also be derived by using the implicit function theorem on the foc of (8.6). It follows immediately from this that if $V''_{t+1}(\mu_{t+1}) = 0$, then $\mathrm{d}x_t^l/\mathrm{d}\mu_t \leq 0$, which in turn implies $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$. Instead suppose that $V''_{t+1}(\mu_{t+1}) < 0$, then the implicit function theorem gives that

$$-\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;=\; \left[-(1-\varphi_L)^2\frac{\mu_t x_t^l}{f_t^3} + \frac{\eta_t^2(1-\eta_t)}{\mu_t(1-\mu_t)}\frac{\mathcal{J}''_0(\eta_t)}{\delta V''_{t+1}(\mu_{t+1})}\right] \Bigg/ \left[(1-\varphi_L)^2\frac{\mu_t^2(1-\mu_t)}{f_t^3} + \frac{\gamma(1-\mu_t)\eta_t^2}{1-\gamma f_t}\frac{\mathcal{J}''_0(\eta_t)}{\delta V''_{t+1}(\mu_{t+1})}\right].$$

If $-\mathrm{d}x_t^l/\mathrm{d}\mu_t \geq 0$, then $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$ follows immediately. Instead suppose that it is negative, in which case the following lower bound can be derived

$$-\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;<\; 0 \quad\Rightarrow\quad -\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;\geq\; \frac{-(1-\varphi_L)^2\mu_t x_t^l/f_t^3}{(1-\varphi_L)^2\mu_t^2(1-\mu_t)/f_t^3} \;=\; -\frac{x_t^l}{\mu_t(1-\mu_t)}.$$

Hence, for $V''_{t+1}(\mu_{t+1}) < 0$ and $-\mathrm{d}x_t^l/\mathrm{d}\mu_t < 0$ I have that

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;\geq\; \frac{1-\varphi_L}{f_t^2}\left(x_t^l - \mu_t(1-\mu_t)\frac{x_t^l}{\mu_t(1-\mu_t)}\right) \;=\; 0\,,$$

which proves the concavity of $V_{t+1}(\mu_t)$ for interior solutions and also gives that $\mathrm{d}\eta_t/\mathrm{d}\mu_t \geq 0$.

Third, suppose that the constrain of the upper bound binds, then

$$l_t^1 > 0 \Rightarrow \begin{cases} x_t^l = 1 \\ l_t^0 = 0 \end{cases} \Rightarrow \begin{cases} \mu_{t+1} = \mu_t + (1-\mu_t)\varphi_L \\ \eta_t = \mu_t \end{cases}.$$

Substituting the above in $V'_t(\mu_t)$, as it is given in (8.7), gives

$$\begin{aligned} V'_t(\mu_t) \;&=\; u_H - u_l + \delta\gamma(1-\varphi_L)V'_{t+1}(\mu_{t+1}) + \delta(1-\gamma)\mathcal{J}'_0(\mu_t) \\ \Rightarrow\;\; V''(\mu_t) \;&=\; \delta\gamma(1-\varphi_L)^2V''_{t+1}(\mu_{t+1}) + \delta(1-\gamma)\mathcal{J}''_0(\mu_t) \;\leq\; 0, \end{aligned}$$

which proves that $V_t(\mu_t)$ is concave in all three possible solutions. □

Proposition 8.1. *The recursive representation $V(\mu_t)$ exists and it is unique. Moreover, $V(\mu_t)$ is twice continuously differentiable and concave. Finally,*

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \geq 0 \quad \textit{and} \quad \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \geq 0.$$

Proof. Let $\mathcal{V}\big([0,1]\big)$ denote the set of bounded, twice continuously differentiable, and concave functions such that $V:[0,1]\to\mathbb{R}$, and consider the operator $T:\mathcal{V}\big([0,1]\big)\to\mathcal{V}\big([0,1]\big)$, given by

$$T(V) \;=\; \Big\{\mu u_H + (1-\mu)u_l + \delta\gamma f V(\mu^*) + \delta(1-\gamma f^*)\mathcal{J}_0(\eta^*)\Big\}$$
$$\text{where} \quad f^* \;=\; \mu + (1-\mu)x^*\,, \quad \eta^* \;=\; \frac{\mu(1-\gamma)}{1-\gamma f^*}\,, \quad \mu^* \;=\; \frac{\mu+(1-\mu)x^*\varphi_L}{f^*}$$

where x^* is the same as the solution for x_t^l derived above when the continuation value was $V_{t+1}(\cdot)$. Then both of Blackwell's sufficient conditions are satisfied. Hence $T(V)$ is a contraction, which implies that it has a unique solution in $\mathcal{V}\big([0,1]\big)$, which is a solution to P_a problem. Finally, because the solution is in $\mathcal{V}\big([0,1]\big)$ all the previous results derived for concave $V_{t+1}(\cdot)$ hold. □

Next, some results are provided for the optimal x_t^l. As a shorthand let $V'(1) = \lim\limits_{\mu_t\to 1} V'(\mu_t)$.

Lemma 8.4.

$$\begin{aligned} V(1) - (1-\varphi_L)V'(1) \;&=\; \frac{1}{1-\delta\gamma(1-\varphi_L)x_t^l(1)}\left(\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l \right.\\ &\left. + \delta\left[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma} + \gamma(1-\varphi_L)(1-x^l(1))\right]\mathcal{J}_0(1) - \delta(1-\varphi_L)[1-\gamma x_t^l(1)]\mathcal{J}_0'(1)\right) \end{aligned} \tag{8.11}$$

Proof. Note that $x^l(\mu_t)$ has to be continuous, as the objective function is twice differentiable in its domain. Then substituting in (8.7) gives

$$\begin{aligned} V'(1) \;=\; & u_H - u_l + \delta\gamma(1-x^l(1))V(1) + \;\delta\gamma(1-\varphi_L)x^l(1)V'(1) \\ & - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\,, \end{aligned}$$

which can be rearranged to get

$$\begin{aligned} V'(1) \;=\; & \frac{1}{1-\delta\gamma(1-\varphi_L)x^l(1)}\Big(u_H - u_l + \delta\gamma(1-x^l(1))V(1) \\ & - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\Big)\,. \end{aligned}$$

Hence substitute the above expression of $V'(1)$ in the right hand side of (8.11) to obtain

$$\begin{aligned} V(1) - (1-\varphi_L)V'(1) \;=\; & \frac{-(1-\varphi_L)}{1-\delta\gamma(1-\varphi_L)x^l(1)} \\ & \times\Big(u_H - u_l - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\Big) + \frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma(1-\varphi_L)x^l(1)}\,V(1)\,. \end{aligned}$$

Finally substitute the functional form of $V(1)$, provided in (8.3), and gather terms to obtain (8.11). ☐

Lemma 8.5. *Continuing a low type is strictly optimal for $\mu_t \to 1$, if*

$$\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1) \;\geq\; 0,. \quad (8.12)$$

If the above inequality is reversed, then stopping is optimal. Finally, if it holds with equality then any $x^l \in [0,1]$ is a solution.

Proof. It follows from (8.6) that

$$\lim_{\mu_t\to 1} \frac{\partial v}{\partial x_t^l}\frac{1/(\delta\gamma)}{1-\mu_t} \;=\; V(1) - (1-\varphi_L)V'(1) - \mathcal{J}_0(1) + \mathcal{J}_0'(1)$$

Hence, substituting the result of the previous lemma gives that

$$\begin{aligned}\lim_{\mu_t\to 1} \frac{\partial v}{\partial x_t^l}\frac{1/(\delta\gamma)}{1-\mu_t} \;=\; & \frac{1}{1-\delta\gamma(1-\varphi_L)x^l(1)} \\ & \times \left(\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1)\right).\end{aligned}$$

The statement follows from noting that the above is the marginal benefit from increasing x^l when $\mu_t \to 1$, the sign of which does not depend on x^l itself. ☐

Proposition 8.2. *Having $x^l(\mu_t) = 0$ for all $\mu_t \in [0,1]$ is suboptimal iff (8.12) holds. In contrast, if it holds in the reversed direction and*

$$\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l + \delta\left[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma} + \gamma(1-\varphi_L)\right]\mathcal{J}_0(1) - \delta(1-\varphi_L)\mathcal{J}_0'(1) \;\leq\; J_0(0)\,, \quad (8.13)$$

then $x_t^l(\mu_t) = 0$ is optimal for all $\mu_t \in [0,1]$. Otherwise, there exists $\tilde{\mu}$ such that $x_t^l(\mu_t) = 0$ is optimal iff $\mu_t > \tilde{\mu}$.

Proof. Suppose the lower bound binds, that is the low type is stopped, then

$$l_t^0 > 0 \;\Rightarrow\; \begin{cases} x_t^l = 0 \\ l_t^1 = 0 \end{cases},$$

hence the foc of (8.6) becomes

$$\frac{-l_t^0}{\delta\gamma(1-\mu_t)} \;=\; V(1) - (1-\varphi_L)V'(1) - \mathcal{J}_0(\eta) + \eta\mathcal{J}_0'(\eta), \quad \text{for} \quad \eta \;=\; \frac{\mu_t(1-\gamma)}{1-\gamma\mu_t}.$$

For this to be a solution it has to be that $l_t^0 \geq 0$, which holds if and only if

$$V(1) - (1-\varphi_L)V'(1) \;\leq\; \mathcal{J}_0(\eta) - \eta\mathcal{J}_0'(\eta). \quad (8.14)$$

Total differentiation on the right hand side of this inequality gives $-\eta\,\mathcal{J}_0''(\eta)\,\frac{\mathrm{d}\eta}{\mathrm{d}\mu_t}$, which is positive. Hence if this inequality is satisfied for any $\mu_t \in [0,1]$, then this is for a convex set $[\tilde{\mu}, 1]$. Finally, note that for $\mu_t = 1$, the above η becomes one, and (8.14) turns into the opposite of (8.12). Hence, if the former is satisfied, then the latter can never hold, which implies that stopping the low type with probability one is never optimal.

In contrast, if (8.14) holds for $\mu_t = 0$

$$V(1) - (1-\varphi_L)V'(1) \;\le\; \mathcal{J}_0(0),$$

then stopping the low type with probability one is optimal for any $\mu_t \in [0,1]$. The left hand side of the above condition, given in (8.11), depends on $x^l(1)$. Despite that, Lemma 8.5 gives that whenever (8.12) is not satisfied, then $x^l(1) = 0$ from which the last statement follows. □

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